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A BOOLEAN ALGEBRA WITH FEW SUBALGEBRAS, INTERVAL BOOLEAN ALGEBRAS AND RETRACTIVENESS

BY

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ABSTRACT. Using $\diamondsuit_{\aleph_1}$ we construct a Boolean algebra B of power $\aleph_1$, with the following properties: (a) B has just $\aleph_1$ subalgebras. (b) Every uncountable subset of B contains a countable independent set, a chain of order type η, and three distinct elements a, b and c, such that $a \cap b = c$. (a) refutes a conjecture of J. D. Monk, (b) answers a question of R. McKenzie. B is embeddable in $P(\omega)$. A variant of the construction yields an almost Jónson Boolean algebra. We prove that every subalgebra of an interval algebra is retractive. This answers affirmatively a conjecture of B. Rotman. Assuming MA or the existence of a Suslin tree we find a retractive BA not embeddable in an interval algebra. This refutes a conjecture of B. Rotman. We prove that an uncountable subalgebra of an interval algebra contains an uncountable chain or an uncountable antichain. Assuming CH we prove that the theory of Boolean algebras in Magidor's and Malitz's language is undecidable. This answers a question of M. Weese.

1. Introduction. In this paper we describe a construction of Boolean algebras (BA's). Our construction yields counterexamples to several questions about BA's. However, we use $\diamondsuit_{\aleph_1}$, so most of the questions remain open in ZFC + CH.

We first construct a BA B of power $\aleph_1$ with the following properties: (1) B has just $\aleph_1$ subalgebras. (2) Every uncountable subset of B contains: a chain of the order type of the rationals, an infinite independent set, and three distinct elements a, b, c such that $a \cap b = c$. (3) B is retractive but is not embeddable in an interval algebra. (See Definitions 1.1 and 1.2.)

Property (1) refutes a conjecture of J. D. Monk, that an infinite BA A has always $2^{|A|}$ subalgebras. In fact, Shelah proved that B has just $\aleph_1$ lower or upper subsemilattices.

R. McKenzie **[Mc]** proved that Monk's conjecture is true if $|A|$ is a strong limit. Let us survey his proof. A subset P of an algebra M is called *irredundant*, if no element $a \in P$ belongs to the subalgebra generated by $P - \{a\}$. Clearly, distinct subsets of an irredundant set generate distinct subalgebras; so if $P \subseteq M$ is irredundant, then M has at least $2^{|P|}$ subalgebras. McKenzie then proved that the subalgebra generated by a maximal irredundant subset of a BA B is dense in B; that is every nonzero element of B is greater than some nonzero element of that subalgebra. Clearly, by Zorn's lemma, every algebra contains a maximal irredundant subset. So

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[1]REMARK. S. Koppelberg proved that any uncountable subset of an interval algebra contains an uncountable irredundant subset. This is still another way to prove 5.7.

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if B is a BA, and $|B|$ is a strong limit, then B contains a maximal irredundant set P, and $|P|$ has to be equal to $|B|$. So B has $2^{|B|}$ subalgebras.

McKenzie then asked whether every infinite BA contains an irredundant set of the same cardinality.

Property (2) refutes this in a strong way. In fact, the detailed formulation of (2) (Theorem 4.6) is the strongest possible in this direction. That is, we divide the countable and finite configurations of subsets of a BA into two classes: $\mathfrak{L}_1$ and $\mathfrak{L}_2$. Every configuration in $\mathfrak{L}_1$ appears as a subset of every uncountable subset of our BA; on the other hand if B is an uncountable BA, there is always an uncountable subset P of B, such that no configuration in $\mathfrak{L}_2$ is realized by a subset of P. (This last fact is trivial, and depends on ZFC.) For example, the configuration: "$a \neq 0 \neq b$ and $a \cap b = 0$" is in $\mathfrak{L}_2$. The configuration: "$\{b_i \mid i \in \omega\}$; $i \neq j \Rightarrow b_i \neq b_j$; $b_0 \neq 0$; and $0 < j < i \Rightarrow b_0 = b_i \cap b_j$ and $1 \neq b_i \cup b_j$" is in $\mathfrak{L}_1$.

DEFINITION 1.1. A BA C is *retractive*, if for every ideal I in C, there is a subalgebra C' of C, such that for every $b \in C$, $|b/I \cap C'| = 1$.

DEFINITION 1.2. Let $\langle I, < \rangle$ be a linear ordering. *The interval algebra based on* I, $B(I)$, is the subalgebra of the power set of I generated by the set $\{V_a \mid a \in I\}$, where $V_a = \{x \mid x \in I \text{ and } x \leqslant a\}$. An *interval algebra* is a BA which is isomorphic to $B(I)$ for some linear ordering $\langle I, < \rangle$.

B. Rotman [**R**, Conjecture A] conjectured that every retractive BA is embeddable in an interval algebra. By property (3) our BA is a strong counterexample to this conjecture, because every subalgebra of our BA which is embeddable in an interval algebra, is of power $\leqslant \aleph_0$. However in §6 we find simpler but weaker counterexamples, assuming either MA or the existence of a Suslin tree.

DEFINITION 1.3. A subset P of a partial ordering is called a *chain* if every two elements of P are comparable. P is called an *antichain* if every two distinct elements of P are incomparable. *Note* that this differs from the more common definition of an antichain. We prove that if B is a subalgebra of an interval algebra, and $|B|$ is regular, then B contains a chain or an antichain of power $|B|$. By property (2), the BA B we construct does not contain uncountable chains or antichains, so it is not embeddable in an interval algebra. On the other hand if $I \subseteq B$ is a dense ideal, then B/I is countable; this property implies retractiveness.

Partial orderings without chains and antichains have been studied extensively. An account of what was done and some open questions can be found in [**DMR**]. The following question which we did not succeed in solving does not appear in [**DMR**].

Question. Does MA + ($\neg$CH) imply that every uncountable BA contains an uncountable chain or an uncountable antichain?

We conjecture that the answer to the above question is negative.

Another question that seems to us interesting and not easy is the following.

Question. Does (CH) imply that there is an uncountable BA B such that every uncountable subset of B contains a triple of distinct elements, a, b, c such that $a \cap b = c$.

We conjecture that also the answer to this question is negative.

DEFINITION 1.4. Let M be an algebra of power κ. M is called *almost Jónson* (AJ), if for every subalgebra N of M of power κ, there is a subset P of M of power $< \kappa$, such that $N \cup P$ generates M.

An algebra M has the *μ-intersection property*, if the intersection of any two subalgebras of M of power μ is of power μ. (M has the μ-IP.)

Notation. If A is a BA let $I(A) = \{a \mid |\{b \mid b \subseteq a\}| \leq \aleph_0\}$.

By changing slightly the construction, we get a BA B of power $\aleph_1$ with the following properties: (1) B is an AJ, $\aleph_1$-IP BA with just $\aleph_1$ lower or upper subsemilattices. (2) $I(B)$ is an AJ lattice, and it is $\aleph_1$-IP lower semilattice.

Notice that these are the best possible results of this kind; in particular, there is no Jónson BA.

In §5 we show that the theory of BA's in Magidor and Malitz's language, L^2, is undecidable. We use the BA's constructed by Bonnet in **[B1]**, thus we assume CH.

This is an answer to a question of M. Weese. Weese **[W]** proved that the theory of BA's in the language L^1 (where $Q^1 x \varphi(x)$ means: there are uncountably many elements satisfying φ) is decidable.

Malitz asked whether there is a first order theory whose set of consequences in L^1 is decidable, but whose set of consequences in L^2 is undecidable. So assuming CH the theory of BA's is such an example; however we believe that CH is not needed and other examples must have been known before.

Let us mention what happens in higher cardinals. Using his omitting type theorem Shelah **[S1]** proved that if $\diamondsuit_{\lambda^-}$ and $\diamondsuit_{\lambda^+}$ hold, then there is a λ-saturated BA B of power λ^+ with the analogous properties. So in property (2) configurations of power λ replace our countable configurations. However, Monk showed that a BA that contains an uncountable independent set is not retractive.

In §6 we make a modest contribution to the question: "When is the free product of two BA's retractive?" This question was raised by B. Rotman. Note that in **[R]** Rotman proved that the free product of an infinite BA and an uncountable BA is not embeddable in an interval algebra.

Our construction resembles Magidor and Malitz's proof of the compactness of $L^{<\omega}$ **[MM]**. Recently Shelah **[S2]** proved a theorem that generalizes our construction; however it does not imply the results presented here, but rather gives general conditions under which such constructions can be carried out.

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2. Notations.

Boolean algebras. A Boolean algebra (BA) is a structure of the form $\langle B, \cup, \cap, -, 0, 1\rangle$. The letters A and B always denote BA's. A, B denote both the Boolean algebra and its universe. $\subseteq$ denotes the partial ordering in a BA; $a \subsetneq b$ means $a \subseteq b$ and $a \neq b$. $a \triangle b$ denotes $(a - b) \cup (b - a)$. When we have to distinguish between the units of different BA's, we denote by $1_B, 0_B$ the 1 and 0 of B. If $a, b \in B$ and $a \cap b = 0$, then a and b are said to be *disjoint*.

At(B) denotes the set of atoms of B. If P is a subset of B, then cl(P) is the subalgebra of B generated by P. A subset P of B is *dense* in B, if for every $b \in B - \{0\}$, there is $c \in P$, $c \neq 0$, such that $c \subseteq b$.

If C and B are BA's, then $C \subseteq B$ always means that C is a subalgebra of B. However, if C is not a BA, then $C \subseteq B$ means that C is a subset of the universe of B.

An ideal I in a BA B is a nonempty subset of B that does not contain 1, is closed under $\cup$, and if $a \in I$ and $B \ni b \subseteq a$ then $b \in I$. If I is an ideal in B, and $a \in B$ then $a/I = \{b \mid b \in B \text{ and } a \triangle b \in I\}$. $B/I = \{b/I \mid b \in B\}$. B/I is regarded as a BA.

If $a \in B - \{0\}$, then $B \restriction a$ is the BA induced by B on the set $\{b \mid B \ni b \subseteq a\}$. If I is an ideal in B, then $I \restriction a = I \cap \{b \mid B \ni b \subseteq a\}$. If $a \notin I$, then $I \restriction a$ is an ideal in $B \restriction a$.

If φ: $B \to A$ is a homomorphism, then $\ker(\varphi) = \{a \mid a \in B \text{ and } \varphi(a) = 0\}$. $\ker(\varphi)$ is an ideal in B.

Partial orderings. If $\langle P, < \rangle$ is any partial ordering, $a, b \in P$, then $(a, b) = \{x \mid x \in P \text{ and } a < x < b\}$, $[a, b] = \{x \mid x \in P \text{ and } a \leqslant x \leqslant b\}$. $(a, b]$ and $[a, b)$ are defined similarly.

If $\langle P, < \rangle$ is a partial ordering $a \in P$ and $D \subseteq P$, then $a < D$ means that for every $d \in D$ $a < d$. $D < a$ is defined similarly.

Sets and models. The cardinality of a set D is denoted by $|D|$. If B is a BA, then $|B|$ denotes the cardinality of the universe of B. If f is a function, then Dom(f), Rng(f) denote the domain and range of f respectively.

If M is a model $\varphi(x_1, \ldots, x_n)$ is a formula in the language of M and $a_1, \ldots, a_n$ belong to the universe of M, then $M \models \varphi[a_1, \ldots, a_n]$ means that $\langle a_1, \ldots, a_n \rangle$ satisfies the formula φ in M.

$M \prec N$ means that M is an elementary submodel of N.

If M is a model, P is a subset of the universe of M, then (M, P) denotes the model gotten from M by adding to the language of M a unary predicate, to represent P.

A Boolean term is a term in the language of Boolean algebras.

3. The construction. We will construct an uncountable BA, in which every nowhere dense set is countable, i.e. of power $\leqslant \aleph_0$. (See Definitions 3.1–3.3.) All the properties mentioned in the introduction will follow from this property.

DEFINITION 3.1. If $a, b \in B$, then $(a, b) \overset{\text{def}}{=} \{c \mid c \in B \text{ and } a \subsetneq c \subsetneq b\}$, and $[a, b] \overset{\text{def}}{=} \{c \mid c \in B \text{ and } a \subseteq c \subseteq b\}$ are called the *open* and respectively the *closed interval* with end points a and b.

DEFINITION 3.2. If $n \geqslant 0$ and $a, b_1, \ldots, b_n, c_1, c_2 \in B$, then $R(a, b_1, \ldots, b_n, c_1, c_2)$ iff $a, b_1, \ldots, b_n$ are pairwise disjoint, $c_1 \subseteq a \cup \bigcup_{i=1}^n b_i$, $a \cup c_1 \subseteq c_2$ and for every $1 \leqslant i \leqslant n$, $(c_2 - c_1) \cap b_i \neq 0$.

DEFINITION 3.3. A subset P of B is called *nowhere dense* (nwd): if for every $n > 0$, $a, b_1, \ldots, b_n \in B$ such that $a, b_1, \ldots, b_n$ are pairwise disjoint and $b_1, \ldots, b_n \neq 0$, *there are* $c_1, c_2 \in B$ such that: $R(a, b_1, \ldots, b_n, c_1, c_2)$ and $P \cap (c_1, c_2) = \varnothing$.

If P is not nwd, then P is called somewhere dense (swd).

Note that the more straightforward way to define nowhere denseness would have been with $n = 1$ only; but it was later noticed by Shelah that with such a definition every uncountable BA would contain an uncountable nwd set.

DEFINITION 3.4. Let $P \subseteq B \subseteq A$. A is *convenient for* P, B (*Notation*: $C(A, P, B)$) if: for every $n \geqslant 0$, $a, b_1, \ldots, b_n \in A$ such that $a, b_1, \ldots, b_n$ are pairwise disjoint and for every $1 \leqslant i \leqslant n$, $b_i \neq 0$, *there are* $c_1, c_2 \in B$ such that: $R(a, b_1, \ldots, b_n, c_1, c_2)$ and $(c_1, c_2) \cap P = \varnothing$.

Note that $C(A, P, A)$ is equivalent to P is a nwd subset of A.

The following lemma summarizes some trivial observations.

LEMMA 3.5. (a) *If* $R(a, b_1, \ldots, b_n, c_1, c_2)$ *and for every* $1 \leqslant i < j \leqslant n$: $b_i \subseteq b'_i$ *and* $b'_i \cap b'_j = b'_i \cap a = 0$, *then* $R(a, b'_1, \ldots, b'_n, c_1, c_2)$.

(b) *If* $P \subseteq B \subseteq A$, $D \subseteq A$ *is dense in* A *and for every* $a \in A$, $n \geqslant 0$, $b_1, \ldots, b_n \in D - \{0\}$ *such that* $a, b_1, \ldots, b_n$ *are pairwise disjoint, there are* $c_1, c_2 \in B$ *such that* $R(a, b_1, \ldots, b_n, c_1, c_2)$ *and* $P \cap (c_1, c_2) = \varnothing$; *then* $C(A, P, B)$.

(c) *If* $\{A_i \mid i < \omega\}$ *is an increasing chain of BA's and for every* $i < \omega$ $C(A_i, P, B)$, *then* $C(\bigcup_{i<\omega} A_i, P, B)$.

If B is a BA and $R = \{a_i = b_i \mid i \in I\}$ is a list of equalities between elements of B, then R determines a homomorphic image of B, namely B/J where J is the ideal generated by the set $\{a_i \triangle b_i \mid i \in I\}$; we will refer to this Boolean algebra as B/R. Let k be the canonical mapping from B to B/R; certainly if $(a = b) \in R$ then $k(a) = k(b)$.

We omit the easy proof of the following lemma.

LEMMA 3.6. *Let* $C = \{0, x, -x, 1\}$ *be a BA with exactly four elements. For every* $i \in I$ *let* $a_i, b_i \in B$, *and assume that for every* $i \in I$, $b_i \subseteq a_i$ *and for every distinct* $i, j \in I$ $a_i \cap a_j = 0$. *Let* B_1 *be the free product of B and C, and let* $R = \{a_i \cap x = b_i \mid i \in I\}$. *Let k be the canonical homomorphism from* B_1 *to* B_1/R; *then* $k \restriction B$ *is* 1-1, *that is, B can be regarded as a subalgebra of* B_1/R.

LEMMA 3.7 (MAIN LEMMA). *Suppose A is countable and atomless. For every* $i < \omega$ *let* $P_i \subseteq B_i \subseteq A$, *and assume* $C(A, P_i, B_i)$, *then there is* A_1 *such that* $A \subsetneqq A_1$, *A is dense in* A_1, *and for every* $i < \omega$ $C(A_1, P_i, B_i)$.

PROOF. Let x be an element not in B, and let $C = \{0, x, -x, 1\}$ be a BA with exactly four elements; we will define by induction a set of equations R in the free product A' of A and C of the form $x \cap a = b$ where $a, b \in A$. A_1 will be A'/R.

Let $T = \{a \cup (b \cap x) \cup (c - x) \mid a, b, c \in A$ and a, b, c are pairwise disjoint$\}$. Note that every element of A' is represented by an element of T, so T will represent (no doubt with repetitions) the elements of A_1. Let $\{s_n \mid n < \omega\}$ be a list of the following objects $A \cup T \cup \{\langle t, b_1, \ldots, b_k, i\rangle \mid t \in T,\ k \geqslant 0,\ b_1, \ldots, b_k$ are pairwise disjoint nonzero elements of A, and $i < \omega\}$. $\{s_n \mid n < \omega\}$ represents the list of tasks that we will have to carry out along the definition of R. So if $s_n \in A$, it will mean that in the nth step we assure that $x \neq s_n$. Taking care of these tasks will ascertain that A_1 will be a proper extension of A. $s_n \in T$ means that in the nth step we have either to decide that $s_n = 0$ or else to find $b \in A$ and to add a relation to R to the effect that $b \subseteq s_n$. This will assure the denseness of A in A_1. When $s_n = \langle t, b_1 \cdots b_k, i\rangle$ we will add a relation for one of the following purposes: (1) make $t \in A$, (2) make $t \cap b_j \neq 0$ for some j, (3) for some $c_1, c_2 \in B_i$ such that $P_i \cap (c_1, c_2) = \varnothing$ make $R(t, b_1, \ldots, b_k, c_1, c_2)$ hold.

The induction hypothesis: after the nth step in the construction we have decided upon the following n relations: $a_i' \cap x = b_i'$ where for every $0 \leqslant i < j \leqslant n$, $b_i' \subseteq a_i'$, $a_i' \cap a_j' = 0$, and $\bigcup_{i=0}^n a_i' \neq 1$. Let $a_n = \bigcup_{i=0}^n a_i'$ and $b_n^* = \bigcup_{i=0}^n b_i'$, then the relation $a_n \cap x = b_n^*$ is equivalent to the set of relations $\{a_i' \cap x = b_i' \mid 0 \leqslant i \leqslant n\}$. So it is equivalent to assume that after n steps we add the relation $a_n \cap x = b_n^*$ where $b_n^* \subseteq a_n \neq 1$.

Step $n + 1$. Suppose $s_n = a \in A$. If $a \cup a_n \neq 1$ let $e \in A$ such that $e \cap (a \cup a_n) = 0$, $e \neq 0$ and $e \cup a \cup a_n \neq 1$. (Remember that A is atomless.) Let us add the relation $x \cap e = e$, so $a_{n+1} = a \cup e$ and $b_{n+1}^* = b_n^* \cup e$, so $a_{n+1} \neq 1$. Since in A'/R $e \neq 0$ this will assure that $x \neq a$. If $a \cup a_n = 1$ then $a - a_n \neq 0$. Let $0 \neq e \subsetneq a - a_n$ and add the relation $x \cap e = 0$. It is clear again that the induction hypothesis holds, and x will be different from a.

Suppose now that $s_n = a \cup (b \cap x) \cup (c - x) = t \in T$. So

$$t = a \cup (b \cap x \cap a_n) \cup (b \cap x - a_n) \cup ((c - x) \cap a_n) \cup ((c - x) - a_n)$$
$$= [a \cup (b \cap b_n^*) \cup ((a_n - b_n^*) \cap c)] \cup [(b - a_n) \cap x] \cup [(c - a_n) - x].$$

So by renaming a, b and c we can w.l.o.g. assume that $(b \cup c) \cap a_n = 0$. Now if $a \neq 0$ then we do not add any relation, because already $a \in A$ and $0 \neq a \subseteq t$. Suppose that $a = 0$. If $b \cup c \neq 0$ we can w.l.o.g. assume that $b \neq 0$. So let $0 \neq e \subsetneq b$ and add the relation $e \cap x = e$; we thus assure that $e \subseteq t$. It is clear that $a_{n+1} = a_n \cup e \neq 1$, and since $e \cap a_n = 0$, the induction hypothesis holds. If $b \cup c = 0$, then $t = 0$, and we do not add any relation to our list.

Suppose now that $s_n = \langle t, b_1, \ldots, b_k, i\rangle$ where $t \in T$, $b_1, \ldots, b_k$ are nonzero pairwise disjoint elements of A (note that k might be 0), and $i \in \omega$. Let $t = a \cup (b \cap x) \cup (c - x)$, and as before w.l.o.g. $(b \cup c) \cap a_n = 0$. If $(\bigcup_{i=1}^k b_i) \cap a \neq 0$ then $t, b_1, \ldots, b_k$ will not be a set of pairwise disjoint elements in A_1, so we do not have to add any relation. Let $d = b \cup c$. If for some $1 \leqslant j \leqslant k$, $b_j \cap d \neq 0$ let us assume w.l.o.g. that $b_j \cap c \neq 0$. Let $0 \neq e \supsetneq b_j \cap c$ and add the relation $x \cap e = 0$. It is easy to check that the induction hypothesis holds and that in A_1 $t \cap b_j \neq 0$, so in A_1 $t, b_1, \ldots, b_k$ will not be pairwise disjoint. If $d = 0$ then $t = a$, and since $C(A, P_i, B_i)$, we do not have to worry. Suppose now that $d \neq 0$ and for every j $d \cap b_j = 0$. So $a, d, b_1, \ldots, b_k$ are pairwise disjoint elements of A and $d, b_1, \ldots, b_k \neq 0$. Since $C(A, P_i, B_i)$, there are $c_1, c_2 \in B_i$ such that $P_i \cap (c_1, c_2) = \varnothing$ and $R(a, d, b_1, \ldots, b_k, c_1, c_2)$. We are going to add a relation in order to make $R(t, b_1, \ldots, b_k, c_1, c_2)$ hold. Let us add the relations $x \cap (c_1 \cap d) = c_1 \cap b$ and $x \cap (d - c_2) = c - c_2$. Since $c_1 \subseteq c_2$, $(c_1 \cap d) \cap (d - c_2) = 0$, and since $(c_2 - c_1) \cap d \neq 0$, $a_{n+1} = a_n \cup (c_1 \cap d) \cup (d - c_2) \neq 1$. Thus the induction hypothesis holds. Let us see that we have assured that $c_1 \subseteq t \cup \bigcup_{i=1}^{k} b_i$ and $t \subseteq c_2$. In order to prove that $c_1 \subseteq t \cup \bigcup_{i=1}^{k} b_i$ it is sufficient to show that $c_1 \cap d \subseteq t$.

$$\begin{aligned} t &\supseteq (b \cap x) \cup (c - x) \supseteq [b \cap c_1 \cap x] \cup [(c - x) \cap c_1] \\ &= (c_1 \cap b) \cup (c \cap c_1 - x) = (c_1 \cap b) \cup [(c \cap c_1) - (c \cap c_1 \cap x)] \\ &\supseteq (c_1 \cap b) \cup [c \cap c_1 - d \cap c_1 \cap x] = (c_1 \cap b) \cup (c \cap c_1 - c_1 \cap b) \\ &= (c_1 \cap b) \cup (c_1 \cap c) = c_1 \cap d. \end{aligned}$$

In order to show that $t \subseteq c_2$ it is sufficient to show that $t \cap d \subseteq c_2 \cap d$.

$$\begin{aligned} t \cap d &= (t \cap d \cap c_2) \cup (t \cap -c_2) \subseteq c_2 \cup [t \cap (d - c_2)] \\ &= c_2 \cup [b \cap x \cap (d - c_2)] \cup [(c - x) \cap (d - c_2)] \\ &= c_2 \cup (b \cap (c - c_2)) \cup [(c - x) \cap (d - c_2)] \\ &= c_2 \cup [c \cap (d - c_2) - x \cap (d - c_2)] = c_2 \cup [c \cap (d - c_2) - (c - c_2)] \\ &= c_2 \cup ((c - c_2) - (c - c_2)) = c_2. \end{aligned}$$

Now it is clear that $R(t, b_1, \ldots, b_k, c_1, c_2)$ will hold in A_1.

Let $R = \{x \cap a_n = b_n^* \mid n < \omega\}$ and let $A_1 = A'/R$. Since by the construction A is dense in A_1, by Lemma 3.5(b) A_1 is as desired. Q.E.D.

THEOREM 3.8 (MAIN THEOREM). ($\diamondsuit_{\aleph_1}$) *There is an uncountable BA B, such that every nwd subset of B is of power $\leqslant \aleph_0$.*

PROOF. We define by induction an increasing continuous chain of BA's $\{B_\alpha \mid \alpha < \aleph_1$ and α is a limit$\}$ and a sequence $\{P_\alpha \mid < \aleph_1$ and α is a limit$\}$ such that: the universe of B_α is α, B_ω is atomless and for every $\omega \leqslant \alpha$ B_ω is dense in B_α; for every α $P_\alpha \subseteq B_\alpha$ and for every $\alpha \leqslant \beta$ $C(B_\beta, P_\alpha, B_\alpha)$.

Let $\{S_\alpha \mid \alpha < \aleph_1\}$ be the sequence assured by $\diamondsuit_{\aleph_1}$. Let B_ω be an atomless BA with universe ω. If δ is a limit of limit ordinals let $B_\delta = \bigcup_{\alpha<\delta} B_\alpha$. Suppose B_α and P_β, $\beta < \alpha$, have been defined. If $C(B_\alpha, S_\alpha, B_\alpha)$ let $P_\alpha = S_\alpha$, and otherwise let $P_\alpha = \varnothing$. Now by the induction hypothesis and by Lemma 3.7, there is a BA $B_{\alpha+\omega}$ with universe $\alpha + \omega$, such that for every $\beta \leqslant \alpha$ $C(B_{\alpha+\omega}, P_\beta, B_\beta)$ and B_α is dense in $B_{\alpha+\omega}$; so the induction hypotheses hold.

Let $B = \bigcup_{\alpha<\aleph_1} B_\alpha$. Suppose by contradiction P is a nwd uncountable subset of B.

Let $F = \{\alpha \mid \alpha < \aleph_1, \alpha$ is a limit and $(B_\alpha, \alpha \cap P) \prec (B, P)\}$; then F is closed and unbounded. Let $S = \{\alpha \mid P \cap \alpha = S_\alpha\}$, then S is stationary; so $S \cap F \neq \varnothing$. Let $\alpha_0 \in S \cap F$. Since P is nwd in B, and $(B_{\alpha_0}, P \cap \alpha_0) \prec (B, P)$, $P \cap \alpha_0$ is nwd in B_{α_0}, thus $P \cap \alpha_0 = P_{\alpha_0}$. Let $a \in P - P_{\alpha_0}$, so there are $c_1, c_2 \in B_{\alpha_0}$, such that $R(a, c_1, c_2)$, i.e. $c_1 \subseteq a \subseteq c_2$, and $P_{\alpha_0} \cap (c_1, c_2) = \varnothing$. But then: $(B_{\alpha_0}, P_{\alpha_0}) \models \forall x(P(x) \to x \notin (c_1, c_2))$, whereas $(B, P) \models P(a) \wedge a \in (c_1, c_2)$. This contradicts the fact that $(B_{\alpha_0}, P_{\alpha_0}) \prec (B, P)$. Q.E.D.

REMARKS. (a) In the construction we can also take care that for every $b \in B - \{0\}$ $|(0, b)| = \aleph_1$. (b) Let us denote by $\prod_{i\in I} B_i$ the full direct product of $\{B_i \mid i \in I\}$ and by $\sum_{i\in I} B_i$ the weakest direct product of $\{b_i \mid i \in I\}$.

Let $\{B_i \mid i \in I\} = \mathfrak{F}$ be a family of BA's. B is called a good product for $\mathfrak{F}$, if there is a countable or finite $J \subseteq I$ such that: $\Sigma_{i\in J} B_i \subseteq B \subseteq \Pi_{i\in J} B_i$, and B/K is countable or finite, where K is the ideal of B generated by $\bigcup_{i\in J} B_i$. The construction can be modified to yield a family $\{B_i \mid i < 2^{\aleph_1}\} \stackrel{\text{def}}{=} \mathfrak{F}$, such that for every $i < 2^{\aleph_1}$ B_i is uncountable, and whenever B is a good product for $\mathfrak{F}$, B does not contain uncountable nwd sets.

Note that this implies that for every $i < j < 2^{\aleph_1}$ and every BA C which is a homomorphic image or embeddable in B_i, and which is a homomorphic image or embeddable in B_j, C is countable or finite.

DEFINITION 3.9. A BA B is called partly concentrated if: (1) B is atomless, and $|B| = \aleph_1$; (2) $I(B)$ is a prime ideal of B; (3) if $P \subseteq B$, $|P| = \aleph_1$, then there are $a \in I(B)$, $b \in B - I(B)$ such that $a \subseteq b$, and for every $a_1 \in I(B)$, $b_1 \in B - I(B)$ such that $a \subseteq a_1 \subseteq b_1 \subseteq b$: $P \cap (a_1, b_1) \neq \varnothing$.

In a method similar to 3.8 one can prove the following theorem.

THEOREM 3.10. ($\diamondsuit_{\aleph_1}$) *There is a partly concentrated BA.*

Theorem 3.11 is due to Shelah; it shows that gap-two theorems are not true for strongly concentrated BA's.

THEOREM 3.11. *Let B have the property that every uncountable set contains distinct elements a, b, c such that $a \cap b = c$; then $|B| \leqslant \aleph_1$.*

PROOF. Let us assume by contradiction that B has the above property but $|B| > \aleph_1$. W.l.o.g. $|B| = \aleph_2$. It is obvious that every uncountable subset of B contains distinct elements a, b, c such that $a \cup b = c$.

We first show that every ideal in B is countably generated. Suppose by contradiction that I is not countably generated. Then there is a sequence $\{a_i \mid i < \aleph_1\} \subseteq I$ such that for every $j_1 < \cdots < j_n < i < \aleph_1$: $a_i \not\subseteq \bigcup_{k=1}^n a_{j_k}$. Let a_i, a_j, a_k be distinct elements in the above sequence such that $a_i \cup a_j = a_k$, $i, j < k$, is certainly impossible. If however $k < j$, then $a_j \subseteq a_k$ contradicts again the property of the sequence. Hence every ideal of B is countably generated.

Let $\{B_i \mid i < \aleph_2\}$ be a strictly increasing continuous sequence of subalgebras of B, such that for every i $|B_i| \leqslant \aleph_1$, and let $a_i \in B_{i+1} - B_i$. Let $C = \{i < \aleph_2 \mid \mathrm{cf}(i) = \aleph_1\}$. For every $i \in C$, let $I_i = \{b \in B_i \mid b \subseteq a_i\}$ and $F_i = \{b \in B_i \mid a_i \subseteq b\}$. Clearly I_i and F_i are respectively an ideal and a filter in B_i, and so they are countably generated. Hence there is $\alpha_i < i$ such that both I_i and F_i are generated by subsets of B_{α_i}.

By Fodor's theorem there is an uncountable set $D \subseteq C$ and $\alpha < \aleph_2$ such that for every $i \in D$ $\alpha_i = \alpha$.

Let i, j, k be distinct elements of D such that $a_i \cup a_j = a_k$. Since $a_i, a_j \subseteq a_k$ there are $b, c \in B_\alpha$ such that $a_i \subseteq b \subseteq a_k$ and $a_j \subseteq c \subseteq a_k$; this can be proved by distinguishing between the cases when $i, j < k$ or $i < k < j$ or $k < i < j$. But then $b \cup c = a_k$ which means that $a_k \in B_\alpha$, a contradiction. Q.E.D.

4. Properties of strongly concentrated BA's.

DEFINITION 4.1. B is called *strongly concentrated* (SC), if $|B| = \aleph_1$, B is atomless, and B does not contain uncountable nwd subsets.

Let $I(B) = \{b \mid b \in B \text{ and } |\{a \mid B \ni a \subseteq b\}| \leqslant \aleph_0\}$.

LEMMA 4.2. *If B is SC, then* $|I(B)| \leqslant \aleph_0$.

PROOF. Suppose by contradiction $|I(B)| = \aleph_1$; then for every $P \subseteq I(B)$: if $|P| \leqslant \aleph_0$, then there is $a \in I(B)$, such that for no $b \in P$, $a \subseteq b$. So $I(B)$ contains a sequence $C = \{c_i \mid i < \aleph_1\}$ such that for every $i < j < \aleph_1$ $c_j \not\subseteq c_i$. Since $|C| = \aleph_1$, it is swd. Let $n > 0$, $a, b_1, \ldots, b_n \in B$ be such that: $a, b_1, \ldots, b_n$ are pairwise disjoint, $b_1, \ldots, b_n \neq 0$, and for every $d_1, d_2 \in B$: if $R(a, b_1, \ldots, b_n, d_1, d_2)$, then $(d_1, d_2) \cap C \neq \varnothing$. For every $1 \leqslant i \leqslant n$ let $\{b_i^j \mid j < \omega\}$ be a sequence such that for every $j < k < \omega$ $b_i \underset{\neq}{\supset} b_i^j \underset{\neq}{\supset} b_i^k$. Let $d_j = a \cup \bigcup_{i=1}^n b_i^j$; then for every $j < \omega$ $R(a, b_1, \ldots, b_n, d_{j+1}, d_j)$. So let $c_{\alpha_j} \in (d_{j+1}, d_j) \cap C$. If $j < k < \omega$, then $c_{\alpha_j} \subseteq c_{\alpha_k}$. On the other hand $\{\alpha_j \mid j < \omega\}$ is not a decreasing sequence, so for some $k > j$ $\alpha_k > \alpha_j$, but by the choice of C, $c_{\alpha_k} \not\subseteq c_{\alpha_j}$, a contradiction. Q.E.D.

THEOREM 4.3. *Suppose B is strongly concentrated, then*:

(a) (*Shelah*) *B has just $2^{\aleph_0}$ lower or upper subsemilattices.*

(b) *Every ideal of B is countably generated* (*as an ideal*). *Every subalgebra of B is generated by an ideal and a countable set.*

(c) *B is retractive*; *that is, if I is an ideal in B, then there is a subalgebra A of B, such that for every $b \in B$ $|A \cap b/I| = 1$.*

(d) (*Shelah*, *Rubin*) *There are just $2^{\aleph_0}$ order preserving functions from B to B.* (f: *$B \to B$ is order preserving, if whenever $a, b \in B$ and $a \subseteq b$; $f(a) \subseteq f(b)$.*)

(e) *If $I(B) = \{0\}$, then B does not contain* 1-1 *or onto endomorphism other than the identity.*

REMARKS. (1) (a) improved the result of the author that B has just $2^{\aleph_0}$ subalgebras. It answers a question of W. Rautenberg.

(2) (d) was first proved by Shelah for BA's that have a property stronger than strong concentration. The author using a similar method proved it for SC BA's. However Bonnet [**B1**] constructed BA's having the properties (d) and (e) assuming CH only.

PROOF. (a) We will prove that every lower subsemilattice of B can be represented as the union of countably many closed intervals. (Note that for every $a \in B$ $\{a\} = [a, a]$ is a closed interval.) Suppose by contradiction that L is a counterexample. Let $\{[a_i, b_i] \mid i < \aleph_1\}$ be an enumeration of all closed intervals which are contained in L. Since L is not a union of $\leqslant \aleph_0$ intervals, we can choose a sequence $C \overset{\text{def}}{=} \{c_i \mid i < \aleph_1\}$ with the following properties: for every $i < \aleph_1$: $c_i \in L - \bigcup_{j<i}[a_j, b_j]$; for every $i < j < \aleph_1$: $c_i \triangle c_j \notin I(B)$, $|C| = \aleph_1$, so there are $n > 0$, $a, b_1, \ldots, b_n \in B$ such that: $a, b_1, \ldots, b_n$ are pairwise disjoint, $b_1, \ldots, b_n \neq 0$; and for every $d_1, d_2 \in B$, if $R(a, b_1, \ldots, b_n, d_1, d_2)$, then $C \cap (d_1, d_2) = \varnothing$. Let $b \in B$

be such that $a \subseteq b$, $b \subseteq a \cup \bigcup_{i=1}^{n} b_i$, and for every $1 \leqslant i \leqslant n$ $b_i - b, b \cap b_i \neq 0$. We will show that $[a, b] \subseteq L$. Let $d \in [a, b]$. For every $1 \leqslant i \leqslant n$, $j = 1,2$, let $d_i^j \subseteq b_i - d$, $d_i^j \neq 0$, and $d_i^1 \cap d_i^2 = 0$. Let $d^j = d \cup \bigcup_{i=1}^{n} d_i^j$, $j = 1,2$. Then $R(a, b_1, \ldots, b_n, d, d^j)$, $j = 1,2$. So let $c^j \in (d, d^j) \cap C$, $j = 1,2$. $d = c^1 \cap c^2$, and $c^1, c^2 \in L$, so $d \in L$.

Since $|[a, b] \cap C| > 1$, $b - a \notin I(B)$; so $|[a, b] \cap C| = \aleph_1$. But $[a, b] \subseteq L$, so for some $i < \aleph_1$ $[a, b] = [a_i, b_i]$, but then $C \cap [a, b] \subseteq \{c_j \mid j < i\}$, so $|C \cap [a, b]| \leqslant \aleph_0$, a contradiction. So every lower subsemilattice of B is the union of countably many closed intervals; so B has $\aleph_1^{\aleph_0} = 2^{\aleph_0}$ lower subsemilattices.

A similar argument holds for upper subsemilattices. Q.E.D.

(b) Let I be an ideal in B; then I is a sublattice of B, so there are $a_i, b_i \in B$, $a_i \subseteq b_i$, $i < \omega$, such that $I = \bigcup_{i<\omega}[a_i, b_i]$, so I is generated as an ideal by the set $\{b_i \mid i < \omega\}$. Let A be a subalgebra of B. Let $A = \bigcup_{i<\omega}[a_i, b_i]$ and $a_i \subseteq b_i$. Let I be the ideal generated by $\{b_i - a_i \mid i < \omega\}$; then A is generated by $I \cup \{b_i \mid i < \omega\}$.

(c) We first prove that if $I \subseteq B$ is a dense ideal, then $|B/I| \leqslant \aleph_0$. If not, let $P \subseteq B$ be such that for every $b \in B$ $|b/I \cap P| = 1$. Since $|P| = \aleph_1$, P is swd. Let $a, b_1, \ldots, b_n$ exemplify this fact. For every $1 \leqslant i \leqslant n$ let $a \neq b_1' \subseteq b_i$ and $b_i' \in I$. So $|P \cap [a, a \cup \bigcup_{i=1}^{n} b_i']| \geqslant 2$, contradicting the choice of P. So $|B/I| \leqslant \aleph_0$.

Secondly the reader can easily check: (∗) if A is a BA, $I \subseteq A$ is an ideal, and $|A/I| \leqslant \aleph_0$, then there is a subalgebra A_1 of A such that for every $a \in A$ $|A_1 \cap a/I| = 1$.

Let I be an ideal in B. Let $J = \{a \mid a \in B$ and for every $b \in I$ $a \cap b = 0\}$; then J is an ideal and $I \cup J$ generates a dense ideal I'. So $|B/I'| \leqslant \aleph_0$. Let A_1 be the subalgebra assured by (∗). Let A_2 be the algebra generated by $J \cup A_1$. We prove that A_2 is as required. If not, there is a such that $0 \neq a \in A_2 \cap I$. W.l.o.g. $a = b \cap c$ where $b \in J$ or $-b \in J$ and $c \in A_1$. But $b \in J$ is impossible, so $-b \in J$; so $c \cap -b \in I'$ and $c \cap b \in I'$, so $c \in I'$. But by choice of A_1, this is impossible. So (c) is proved.

(d) Let us first mention that an SC BA does not contain uncountable antichains. (See Theorem 4.6(c).)

Let f: $B \to B$ be an order preserving function. We will prove: (∗) there is a sequence $\{\langle a_i, b_i, c_i, d_i\rangle \mid i < \omega\}$ such that: $B = \bigcup_{i<\omega}[a_i, b_i]$, and for every $x \in [a_i, b_i]$, $f(x) = c_i \cup (d_i \cap x)$.

Suppose f is a counterexample to (∗). In a way similar to what was done in part (a), one can choose an uncountable subset C of$B$ with the following properties: (1) If $a, b, c, d \in B$ are such that for every $x \in [a, b]$, $f(x) = c \cup (d \cap x)$; then $|C \cap [a, b]| \leqslant \aleph_0$; (2) if $c_1, c_2 \in C$ and $c_1 \neq c_2$, then $c_1 \triangle c_2 \notin I(B)$.

Let $a, b_1, \ldots, b_n$ exemplify the fact that C is swd. We will first find $a', b_1', \ldots, b_n'$ and $\sigma_1, \sigma_2, \sigma_3 \subseteq \{1, \ldots, n\}$ such that: $a \subseteq a' \subseteq a \cup \bigcup_{i=1}^{n} b_i$, $0 \neq b_i' \subseteq b_i$, $b_1' \cap a' = 0$, $\sigma_1 \cup \sigma_2 \cup \sigma_3 = \{1, \ldots, n\}$, and for every $x \in [a', a' \cup \bigcup_{i=1}^{n} b_i]$ and $1 \leqslant i \leqslant n$: if $i \in \sigma_1$ then $f(x) \supseteq b_i'$, if $i \in \sigma_2$ $f(x) \cap b_i' = 0$, and if $i \in \sigma_3$ then $f(x) \cap b_i' = x \cap b_i'$.

Let $\sigma_1 \subseteq \{1, \ldots, n\}$ be a maximal set with the property that there is $c \in B$ such that: $a \subseteq c \subseteq a \cup \bigcup_{i=1}^{n} b_i$, for every $1 \leqslant i \leqslant n$ $b_i \cap c \neq 0 \neq b_i - c$, and for every $i \in \sigma_1$ $(f(c) - c) \cap b_i \neq 0$. For every $i \in \sigma_1$ let $b_i^1 = (f(c) - c) \cap b_i$; for every $i \in \{1, \ldots, n\} - \sigma_1$ let $b_i^1 = b_i - c$. Let $\sigma_2 \subseteq \{1, \ldots, n\}$ be a maximal set with the property: there is $c_1 \in B$ such that $c_1 \in [c, c \cup \bigcup_{i=1}^n b_i^1]$, for every $i \in \{1,\ldots,n\}$ $c_1 \cap b_i^1 \neq 0$, and for every $i \in \sigma_2$, $f(c_1) \cap b_i^1 \subsetneq c_1 \cap b_i^1$. Let $a' = c \cup (f(c_1) \cap \bigcup_{i\in\sigma_2} b_i^1)$, and let $b_i' = (c_1 - a') \cap b_i^1$, $i = 1,\ldots,n$.

It is easily seen that $a', b',\ldots,b_n', \sigma_1, \sigma_2$ and $\sigma_3 \overset{\text{def}}{=} \{1,\ldots,n\} - \sigma_1 - \sigma_2$ are as desired.

It is also clear that $a', b_1',\ldots,b_n'$ exemplify the fact that C is swd. So let us rename $a', b_1',\ldots,b_n'$ by $a, b_1,\ldots,b_n$ respectively. Let $C_1 = C \cap [a, a \cap \bigcup_{i=1}^n b_i]$. Suppose first (**): $|\{f(c) - \bigcup_{i=1}^n b_i \mid c \in C_1\}| = \aleph_1$. Then there is a $C_2 \subseteq C_1$ such that $|C_2| = \aleph_1$, and for every $c_1, c_2 \in C_2$: if $c_1 \neq c_2$, then $f(c_1) - \bigcup_{i=1}^n b_i \neq f(c_2) - \bigcup_{i=1}^n b_i$. Let $P = \{(c \cap \bigcup_{i=1}^n b_i) \cup (-f(c) - \bigcup_{i=1}^n b_i) \mid, c \in C_2\}$; then P is an uncountable antichain. But this is impossible, so (**) does not hold. So there is $C_3 \subseteq C_1$ such that $|C_3| = \aleph_1$ and for every $c_1, c_2 \in C_3$: $f(c_1) - \bigcup_{i=1}^n b_i = f(c_2) - \bigcup_{i=1}^n b_i$. Let $e_1 = \bigcup_{i\in\sigma_1} b_i$, $e_3 = \bigcup_{i\in\sigma_3} b_i$, $c_0 \in C_3$ and $d = f(c_0) - \bigcup_{i=1}^n b_i$; then, for every $x \in C_3$, $f(x) = (e_1 \cup d) \cup (e_3 \cap x)$. Let $a', b_1',\ldots,b_k'$ exemplify the fact that C_3 is swd. It is easy to see that if $x \in (a', a' \cup \bigcup_{i=1}^k b_i')$ and for every $1 \leqslant i \leqslant k$, $x \cap b_i' \neq 0 \neq b_i' - x$, then $f(x) = (e_1 \cup d) \cup (e_3 \cap x)$. Let $0 \neq b_i^1 \subsetneq b_i^2 \subsetneq b_i'$, $i = 1,\ldots,k$, and let $a'' = a' \cup \bigcup_{i=1}^k b_i^1$ and $b'' = \bigcup_{i=1}^k (b_i^2 - b_i^1)$; then, for every $x \in [a'', a'' \cup b'']$, $f(x) = (e_1 \cup d) \cup (e_3 \cap x)$ and $|C_3 \cap [a'', b'']| = \aleph_1$. This contradicts the choice of C. So we proved (*). It follows trivially that $|\{f \mid f: B \to B$ is order preserving$\}| \leqslant 2^{\aleph_0}$.

Question. For which formulas $\varphi(x_1,\ldots,x_n)$ are there just $2^{\aleph_0}$ φ preserving functions?

(e) REMARKS. (1) Baumgartner noted that if B does not contain uncountable antichains and $I(B) = \{0\}$ then B is rigid. We note in a similar way, that such a B does not have 1-1 order preserving functions other than the identity.

(2) Every BA with more than two elements has an onto order preserving function different from the identity. For, certainly it is true for finite BA's; so let $|B| \geqslant \aleph_0$, let $\{a_i \mid i < \omega\}$ be a set of pairwise disjoint nonzero elements of B. Define $f(x) = x$, if $\{i \mid x \cap a_i \neq 0\}$ is infinite; otherwise define $f(x) = x - a_i$ where $i = \max(\{j \mid a_j \cap x \neq 0\})$. f is as desired. Note also that f preserves disjointness.

(3) Bonnet **[B1]** proved that if B is an interval algebra without 1-1 endomorphisms except the identity, then B does not have onto endomorphisms except the identity. It was noticed by Monk and Loats that this fact is true for every retractive BA. In fact it is true for every retractive algebra.

PROOF OF (e). Suppose $f: B \to B$ is 1-1 and order preserving. Assume by contradiction $f(a) \neq a$.

Case 1: $a - f(a) \neq 0$. Let $P = \{x \cup [f(a) - f(x)] \mid x \subseteq a - f(a)\}$, then P is an uncountable antichain. By Theorem 4.6(c) this is a contradiction.

Case 2: $f(a) \supsetneq a$. In this case $\{x \cup (-f(x) \cap -f(a)) \mid x \in (a, f(a))\}$ is an uncountable antichain, and again we reach a contradiction.

Suppose B is retractive, and does not have 1-1 endomorphism except the identity. Let f be an onto endomorphism and $I = \ker(f)$, let $A \subseteq B$ be a subalgebra of B such that for every $b \in B$, $|A \cap b/I| = 1$ then $f \upharpoonright A$ is an isomorphism between A

and B. So, $(f\restriction A)^{-1}$ is a 1-1 endomorphism from B to B. So $(f\restriction A)^{-1} = \mathrm{Id}$, hence $f = \mathrm{Id}$. Q.E.D.

If B is a BA and $\varphi(x_1,\ldots,x_n)$ is a formula in the language of BA's, let $B\restriction\varphi$ be the structure $M \overset{\mathrm{def}}{=} \langle A, R^M\rangle$ where A is the universe of B and $R^M = \{\langle b_1,\ldots,b_n\rangle \mid B \models \varphi[a_1,\ldots,a_n]\}$.

Questions. Suppose B is SC. (1) For which formulas φ $B\restriction\varphi$ is retractive.

(2) For which φ's $B\restriction\varphi$ does not have 1-1 endomorphisms other than the identity, and for which φ's $B\restriction\varphi$ does not have onto endomorphisms other than the identity.

DEFINITION 4.4. A *configuration* is a quantifier free, consistent complete (in the set of quantifier free formulas) type in the language of BA's, and with variables $\{x_i \mid i \in I\}$ where $|I| \leqslant \aleph_0$.

Let B be a BA, L a configuration and $P \subseteq B$; *L appears in P*, if for some $\{a_i \mid I < \alpha\} \subseteq P$ $\{a_i \mid i < \alpha\}$ realizes L. If L does not appear in P, then we say that P is *L-free*.

L is called a *good configuration* if there is a linear ordering $<$ of I, such that for every $i_1 < i_2 < \cdots < i_k \in I$ and a term $\tau(x_1,\ldots,x_{k-1})$: $(\bigcup_{j=1}^k x_{i_j} = 1) \notin L$, $(\bigcap_{j=1}^k x_{i_j} = 0) \notin L$ and $(\tau(x_{i_1},\ldots,x_{i_{k-1}}) = x_{i_k}) \notin L$.

The reader can easily ascertain the following observation.

Observation 4.5. For every uncountable BA B there is an uncountable subset P of B, such that every configuration that appears in P is good.

EXAMPLES. The following configurations are good: (1) $\{x_i \mid i < \omega\}$ is an independent set; (2) $\{x_r \subsetneq x_q \mid r, q$ are rationals and $r < q\}$; (3) $x_1 = x_2 \cap x_3$, $x_2 \neq x_3 \neq x_1 \neq 0$ and $x_2 \cup x_3 \neq 1$.

THEOREM 4.6. (a) *Let A be an atomless BA, L be a good configuration with order type $\leqslant \omega$ and $P \subseteq A$ be an L-free subset of A; then P is nwd.* (b) *If B is SC, L is a good configuration with order type $\leqslant \omega$ and $P \subseteq B$ is uncountable, then L appears in P.* (c) *If B is SC, then B does not contain uncountable chains or antichains.*

PROOF. (b) is a trivial corollary of A, and (c) is a trivial corollary of (b).

PROOF OF (a). We prove that if $P \subseteq A$ is swd, and L is a good configuration then L appears in P. W.l.o.g., L is in the variables $\{x_i \mid i < \omega\}$. Let $a, b_1,\ldots,b_m$ exemplify the fact that P is swd. We now define by induction a sequence $\{p_i \mid i < \omega\} \subseteq P$ with the following induction hypotheses: Suppose $p_0,\ldots,p_{n-1}$ have been defined $n > 0$, and let $\{r_1,\ldots,r_l\} = \mathrm{At}(\mathrm{cl}(\{p_0,\ldots,p_{n-1}\}))$; *then*:

(1) for every i, $0 \leqslant i < n$: $a \subseteq p_i \subseteq a \cup \bigcup_{i=1}^m b_i$;

(2) for every i, $0 \leqslant i < n$, and for every j, $1 \leqslant j \leqslant m$: $r_i \cap b_j \neq 0$;

(3) $\langle p_0,\ldots,p_{n-1}\rangle$ realizes $L\restriction\{x_i \mid i < n\}$.

We first construct p_0. Let $p_0 \in P \cap [a, a \cup \bigcup_{i=1}^m b_i]$ be such that for every $1 \leqslant i \leqslant m$, $b_i \cap p_0 \neq 0 \neq b_i - p_0$. Clearly, the induction hypotheses for $n = 1$ are satisfied.

Suppose $p_0,\ldots,p_{n-1}$ have been defined, and $n > 0$. Let $\{r_1,\ldots,r_l\}$ be as above. W.l.o.g. $r_1 = \bigcap_{i<n} p_i$, and $r_l = \bigcap_{i<n} - p_i$. For every $1 \leqslant i \leqslant l$, let $r_i = \tau_i(p_0,\ldots,p_{n-1})$, and let us denote the term $\tau_i(x_0,\ldots,x_{n-1})$ by y_i. Let $\sigma_1 = \{i \mid (y_i \subseteq x_n) \in L\}$, $\sigma_2 = \{i \mid (y_i \cap x_n = 0) \in L\}$ and $\sigma_3 = \{i \mid (y_i \cap x_n \neq 0 \neq y_i - x_n) \in L\}$.

Since for no term τ, $(\tau(x_0,\ldots,x_{n-1}) = x_n) \in L$, $\sigma_3 \neq \varnothing$. Since $(\bigcap_{i\leqslant n} x_i \neq 0) \in L$, $1 \notin \sigma_2$; and since $(\bigcup_{i\leqslant n} x_i \neq 1) \in L$, $l \notin \sigma_1$.

For every $i \in \sigma_3$ and $1 \leqslant j \leqslant m$, let $0 \neq d_{ij}^1 \subsetneq d_{ij}^2 \subsetneq r_i \cap b_j$. Let

$$c_1^n = a \cup \left(\bigcup_{i\in\sigma_1} r_i \right) \cup \bigcup \{ d_{ij}^1 \mid i \in \sigma_3 \text{ and } 1 \leqslant j \leqslant m \},$$

and

$$c_2^n = a \cup \left(\bigcup_{i\in\sigma_1} r_i \right) \cup \bigcup \{ d_{ij}^2 \mid i \in \sigma_3 \text{ and } 1 \leqslant j \leqslant m \}.$$

We first show that $R(a, b_1,\ldots,b_m, c_1^n, c_2^n)$ holds. In order to show that $c_1^n \subseteq a \cup \bigcup_{i=1}^m b_i$, it suffices to show that for every $i \in \sigma_1$, $r_i \subseteq a \cup \bigcup_{i=1}^m b_i$. But since $l \notin \sigma_1$, there is at least one j, $0 \leqslant j < n$, such that $(y_i \subseteq x_j) \in L$. So $r_i \subseteq p_j \subseteq a \cup \bigcup_{t=1}^m b_t$. Certainly $c_2^n \subseteq c_1^n \cup a$. $\sigma_3 \neq \varnothing$, so let $\beta \in \sigma_3$. Let $1 \leqslant j \leqslant m$; then $(c_2^n - c_1^n) \cap b_j \supseteq d_{\beta j}^2 \neq 0$. So we proved that $R(a, b_1,\ldots,b_m, c_1^n, c_2^n)$ holds.

Let $p_n \in P \cap [c_1^n, c_2^n]$. Since $a \subseteq c_1^n \subseteq p_n \subseteq c_2^n \subseteq a \cup \bigcup_{i=1}^m b_i$, condition (1) of the induction hypotheses holds. If $r \in \mathrm{At}(\mathrm{cl}(\{p_0,\ldots,p_n\}))$, then either: $r \in \{r_1,\ldots,r_l\}$ and then for every $1 \leqslant j \leqslant m$ $r \cap b_j \neq 0$; or else there is $\beta \in \sigma_3$, such that $r \supseteq \bigcup_{j=1}^m d_{\beta j}^1$, or $r \supseteq \bigcup_{j=1}^m (r_\beta \cap b_j - d_{\beta j}^2)$, so in both cases for every $1 \leqslant j \leqslant m$, $r \cap b_j \neq 0$. Hence, condition (2) of the induction hypotheses holds. It is clear that for every $1 \leqslant i \leqslant l$: $p_n \supseteq r_i$ iff $(x_n \supseteq y_i) \in L$, and $r_i \cap p_n \neq 0 \neq r_i - p_n$ iff $(y_i \cap x_n \neq 0 \neq y_i - x_n) \in L$; so $\langle p_0,\ldots,p_n \rangle$ realizes $L \restriction \{x_0,\ldots,x_n\}$. Thus (a) has been proved.

REMARK. Remember that by [**BaK**], and SC BA is embeddable in $\langle P(\omega), \cup, \cap, -, \varnothing, \omega \rangle$.

Question. Let B be SC and $P \subseteq B$ be uncountable. Does every good configuration appear in P?

THEOREM 4.7. *Let B be a partly concentrated BA. Then*: (1) *B is an AJ, $\aleph_1$-IP BA with just $\aleph_1$ lower or upper subsemilattices.*

(2) *$I(B)$ is an AJ lattice and it is $\aleph_1$-IP lower semilattice.*

PROOF. Similar to the arguments previously presented in this section.

5. Some facts about interval algebras. If $\langle J, < \rangle$ is a linear ordering let $J^+ = J \cup \{-\infty, \infty\}$ (we assume that $-\infty, \infty \notin J$); we define the ordering on J^+ in the obvious way. If $a, b \in J^+$ let $(a,b] = \{x \mid x \in J \text{ and } a < x \leqslant b\}$. Note that $-\infty, \infty \notin (a,b]$. Every element a of the interval algebra $B(J)$ (see Definition 1.2) can be uniquely represented in the form $\bigcup_{i=1}^n (a_i,b_i]$, where $n \geqslant 0$, $a_1, b_1,\ldots,a_n, b_n \in J^+$, and $-\infty \leqslant a_1 < b_1 < a_2 < \cdots < a_n < b_n \leqslant \infty$. We call this representation the canonical representation of a. We denote $\sigma_a = \{a_1, b_1,\ldots,a_n, b_n\}$. Note that $\sigma_{a-b}, \sigma_{a\cap b}, \sigma_{a\cup b} \subseteq \sigma_a \cup \sigma_b$. Let $\vec{\sigma}_a = \langle a_1, b_1,\ldots,a_n, b_n \rangle$. We make the convention that $\mathrm{Dom}(\vec{\sigma}_a) = \{1,\ldots,2n\}$.

Note that if $\langle J, < \rangle$ is a linear ordering and $\varnothing \neq J_1 \subseteq J$, then $B(J_1)$ can be embedded in $B(J)$ in a natural way; so we regard $B(J_1)$ as a subalgebra of $B(J)$.

The following theorem answers affirmatively a question of B. Rotman [**R**, Conjecture B].

THEOREM 5.1. *Every subalgebra of an interval algebra is retractive.*

PROOF. We prove the following statement which is equivalent to the theorem. If $\langle J, < \rangle$ is a linear ordering, $B = B(J)$ is its interval algebra. $I \subseteq B$ is an ideal, and $A \subseteq B$ is a subalgebra of B; then there is a subalgebra $A' \subseteq A$, such that for every $a \in A \mid A' \cap a/I \mid = 1$.

Let $J_0 \subseteq J^+$ be a maximal set with the property that $B(J_0) \cap A \cap I = \{0\}$. We will show that for every $a \in A$, $\mid B(J_0) \cap A \cap a/I \mid = 1$, so A' can be chosen to be $B(J_0) \cap A$.

For every $a \in B$ let $\bar{\sigma}_a = \sigma_a - J_0$. $\bar{\sigma}_a = \varnothing$ iff $a \in B(J_0)$. We will prove if $a \in A$ and $\bar{\sigma}_a \neq \varnothing$, then there is $a' \in A$ such that $a/I = a'/I$ and $\mid \bar{\sigma}_{a'} \mid < \mid \bar{\sigma}_a \mid$. Let $a \in A$, $a = \bigcup_{i=1}^{n} (a_i, b_i]$ be its canonical representation and suppose $a_k \notin J_0$. So $B(J_0 \cup \{a_k\}) \cap A \cap I \neq \{0\}$. Let $0 \neq b \in B(J_0 \cup \{a_k\}) \cap A \cap I$, and $b = \bigcup_{i=1}^{m} (c_i, d_i]$ be its natural representation. For some $1 \leqslant j \leqslant m$ either: (1) $a_k = c_j$, or (2) $a_k = d_j$, and $\sigma_b - \{a_k\} \subseteq J_0$.

If (1) happens, then $\sigma_{a-b} \not\ni a_k$; however, $\sigma_{a-b} \subseteq \sigma_a \cup \sigma_b$. Since $\sigma_b - \{a_k\} \subseteq J_0$ $\mid \bar{\sigma}_{a-b} \mid < \mid \bar{\sigma}_a \mid$. If (2) happens, then $\sigma_{a \cup b} \not\ni a_k$, so $\mid \bar{\sigma}_{a \cup b} \mid < \mid \bar{\sigma}_a \mid$. Since $b \in I$, $a/I = a - b/I = a \cup b/I$, and since $b \in A$, $a - b$, $a \cup b \in A$. A similar argument holds when for some k $b_k \in \bar{\sigma}_a$.

So, by descending induction, for every $a \in A$ there is $a' \in A \cap B(J_0)$ such that $a/I = a'/I$. Since $B(J_0) \cap A \cap I = \{0\}$, for every $a \in A$ $\mid B(J_0) \cap A \cap a/I \mid \leqslant 1$. So $A' \overset{\text{def}}{=} B(J_0) \cap A$ is as desired. Q.E.D.

Let $B_1 \times B_2$ be the direct product of B_1 and B_2. A Boolean term $\tau(x_1, \ldots, x_n)$ is called *trivial* if the value of τ under every assignment into a BA is 0. *Otherwise τ is said to be nontrivial.*

An n-tuple $\langle a_1, \ldots, a_n \rangle$ of elements of a BA B is *dependent* if for some nontrivial $\tau(x_1, \ldots, x_n)$: $\tau(a_1, \ldots, a_n) = 0$.

Let $\langle J, < \rangle$ be a linear ordering $B = B(J)$, $a \in B$ and $\mid \sigma_a \mid = k$. If $\vec{\sigma}_a(1) > -\infty$, we define $\vec{\sigma}_a(0) = -\infty$; if $\vec{\sigma}_a(k) < \infty$, we define $\vec{\sigma}_a(k+1) = \infty$. So, $\{1, \ldots, k\} \subseteq \operatorname{Dom}(\vec{\sigma}_a) \subseteq \{0, \ldots, k+1\}$.

DEFINITION 5.2. Let $\langle J, < \rangle$ be a linear ordering, $B = B(J)$, $\vec{a} = \{a_i \mid i < \alpha\} \subseteq B$. $\vec{a}$ is called homogeneous if: (1) there is k such that for every $i < \alpha \mid \sigma_{a_i} \mid = k$; and (2) for every $1 < j < \alpha$: $\sigma_{a_i} \cap \{-\infty, \infty\} = \sigma_{a_j} \cap \{-\infty, \infty\}$, $\sigma_{a_i} \cap \sigma_{a_j} - \{-\infty, \infty\} = \varnothing$, and there is $l = l_{ij}^{\vec{a}} < \max(\operatorname{Dom}(\vec{\sigma}_{a_i}))$, $l \in \operatorname{Dom}(\vec{\sigma}_{a_i})$, such that $\vec{\sigma}_{a_i}(l) < \infty$, and $\vec{\sigma}_{a_i}(l) < \sigma_{a_j} - \{-\infty, \infty\} < \vec{\sigma}_{a_i}(l+1)$.

Note that $\vec{a}$ might be the constant sequence of 0's or the constant sequence of 1's.

LEMMA 5.3. *There is $n_0 < \omega$ such that for every homogeneous $\vec{a}$, every sequence of n_0 elements of $\vec{a}$ is dependent.*

PROOF. Easy to check.

LEMMA 5.4. *Let J and B be as in 5.2, and let $P \subseteq B$, $\mid P \mid = \lambda > \aleph_0$, and λ is regular; then there is a 1-1 sequence $\{a_i \mid i < \lambda\} \subseteq P$, $n < \omega$ and $-\infty = b_0 < b_1 < \cdots < b_n = \infty$, such that for every $m < n$ $\{a_i \cap (b_m, b_{m+1}] \mid i < \lambda\}$ is homogeneous in $B \upharpoonright (b_m, b_{m+1}]$.*

PROOF. By a simple cleaning process.

The following theorem is a weakened version of Theorem 2(b) in **[Ru]** which had an error. At present we do not know to prove or disprove it.

THEOREM 5.5. *Let J, B, P, and λ be as in 5.4. Then there is $m < \omega$ and a 1-1 $\vec{a} = \{a_i \mid i < \lambda\} \subseteq P$, such that every m elements of $\vec{a}$ are dependent.*

PROOF. Let n, $b_0, \ldots, b_n$ and $\vec{a} = \{a_i \mid i < \lambda\}$ be as in 5.4 and let $m = n \cdot n_0$. (n_0 was defined in 5.3.)

Let $c_1^1, \ldots, c_{n_0}^1, c_1^2, \ldots, c_{n_0}^2, \ldots, c_1^n, \ldots, c_{n_0}^n$ be elements of $\vec{a}$. By 5.3 for every $i < n$, there is a nontrivial term $\tau_i(x_1, \ldots, x_{n_0})$, such that in $B \upharpoonright (b_i, b_{i+1}]$:

$$\tau_i\left(c_1^i \cap (b_i, b_{i+1}], \ldots, c_{n_0}^i \cap (b_i, b_{i+1}]\right) = 0.$$

Let $\tau(x_1^1, \ldots, x_{n_0}^1, \ldots, x_1^n, \ldots, x_{n_0}^n) = \bigcap_{i=1}^n \tau_i(x_1^i, \ldots, x_{n_0}^i)$; then τ is nontrivial and

$$\tau\left(c_1^1, \ldots, c_{n_0}^1, \ldots, c_1^n, \ldots, c_{n_0}^n\right) = 0. \quad \text{Q.E.D.}$$

An object $C = \langle \{a_i \mid i < \lambda\}, \langle b_0, \ldots, b_n \rangle \rangle$ which satisfies the conclusion of 5.4 is called a system.

If $\vec{a} = \{a_i \mid i < \lambda\} \subseteq B$ is homogeneous, let $m(\vec{a}, B) = |\sigma(a_i) - \{-\infty, \infty\}|$, where i is any ordinal $< \lambda$. If C is as above, we define

$$m(C) = 2 \cdot \sum_{j<n} m\left(\left\{a_i \cap \left(b_j, b_{j+1}\right] \mid i < \lambda\right\}, B \upharpoonright \left(b_j, b_{j+1}\right]\right)$$

$$-\left|\left\{j \mid \left\{a_i \cap \left(b_j, b_{j+1}\right] \mid i < \lambda\right\} \text{ is not a constant sequence}\right\}\right|.$$

Let us now quote Theorem 3 from **[BaK]**.

If λ is a regular cardinal and B does not contain an antichain of power λ, then B contains a dense subset of power $< \lambda$.

THEOREM 5.6. *Let $\langle J, < \rangle$ be a linear ordering, and let $B \subseteq B(J)$ be of power λ, where λ is regular. Then B contains a chain or an antichain of power λ.*

PROOF. Suppose $|B| = \lambda$, and B does not contain antichains of power λ. Let $C = \langle \vec{a}, \langle b_0, \ldots, b_n \rangle \rangle$ be a system, where $\vec{a}$ is 1-1 and has length λ, and $m(C)$ is minimal. We will show that for some $D \subseteq \lambda$, $|D| = \lambda$ and $\{a_\alpha \mid \alpha \in D\}$ is a chain. Let $\vec{a} = \{a_j \mid j < \lambda\}$. For every $i < n$, let $a_j^i = a_j \cap (b_i, b_{i+1}]$, $\vec{a}^i = \{a_j^i \mid j < \lambda\}$, and $B^i = B \upharpoonright (b_i, b_{i+1}]$. Note that since B does not have antichains of power λ, neither does B^i. We shall first show that it can be assumed that for every $i < n$:

(∗) for every $j < k < \lambda$, a_j^i and a_k^i are comparable.

Let $m_i = |\sigma_{a_j^i} - \{-\infty, \infty\}|$. (∗) certainly holds for i, if $|m_i| \leqslant 1$. Suppose $m_i > 1$. By duality, we can w.l.o.g. assume that $-\infty \notin \sigma_{a_j^i}$. Let $D_i = \{j \mid$ there is $k > j$ such that $1 \leqslant l_{jk}^{\vec{a}^i} \leqslant m_i - 1\}$. ($l_{jk}^{\vec{a}}$ was defined in 5.2.) Suppose $|D_i| = \lambda$, for every $j \in D_i$, let $k(j)$ be as assured in the definition of D_i. For every $j \in D_i$, let $c_j = a_j^i - a_{k(j)}^i$. Let $E \subseteq B^i - \{0\}$ be dense in B^i and $|E| < \lambda$. Since for every $j \in D_i$, $c_j \neq 0$, there is $e_j \in E$ such that $e_j \subseteq c_j$. Since λ is regular, there is $e \in E$ such that: $D' \overset{\text{def}}{=} \{j \mid e_j = e\}$ has power λ. Let $t = \min(\sigma_e)$; then for every $j \in D'$, $\vec{\sigma}_{a_j^i}(1) < t < \vec{\sigma}_{a_j^i}(m_i)$. But then there is l, $1 \leqslant l < m_i$ such that $D'' \stackrel{\text{def}}{=} \{j \mid \vec{\sigma}_{a_j^i}(l) < t < \vec{\sigma}_{a_j^i}(l+1)\}$ has power λ. Let

$$C' = \langle \{a_j \mid j \in D''\}, \langle b_0, \ldots, b_i, t, b_{i+1}, \ldots, b_n \rangle \rangle;$$

then $m(C') = m(C) - 1$, a contradiction.

So $|D_i| < \lambda$. By deleting $\{a_j \mid j \in \bigcup\{D_i \mid m_i > 1\}\}$ from $\{a_j \mid j < \lambda\}$, we can assume that for every $i < n$ if $m_i > 1$, then $D_i = \varnothing$.

It is easy to check that if $D_i = \varnothing$, then either: for every $j < k < \lambda$, a_j^i and a_k^i are comparable; or for every $j < k < \lambda$ a_j^i and a_k^i are incomparable. Since the second case cannot hold, $(*)$ is proved. For every $i, j < n$ and for every $k < \lambda$, let $D_{ijk} = \{\alpha \mid \alpha < \lambda,\ a_\alpha^i \subsetneqq a_k^i \text{ and } a_k^j \subsetneqq a_\alpha^j\}$. We prove that for every i, j, k as above, $|D_{ijk}| < \lambda$. If not, let i, j, k be a counterexample. For every $\alpha \in D_{ijk}$ let $b^\alpha = a_\alpha \cap a_k$, so $b^\alpha \cap (b_j, b_{j+1}] = a_k^j$, and $b^\alpha \cap (b_i, b_{i+1}] = a_\alpha^i$. Thus $\{b^\alpha \mid \alpha \in D_{ijk}\}$ is 1-1. Let $\vec{c}$ be a subsequence of length λ of $\{b^\alpha \mid \alpha \in D_{ijk}\}$, such that for some $b'_0, \ldots, b'_{n'}$, $C' \stackrel{\text{def}}{=} \langle \vec{c}, \langle b'_0, \ldots, b'_{n'} \rangle \rangle$ is a system; then $m(C') < m(C)$, a contradiction, so $|D_{ijk}| < \lambda$.

We now define by induction $D \subseteq \lambda$, $|D| = \lambda$, such that $\{a_\alpha \mid \alpha \in D\}$ is a chain. Suppose α_ν have been defined for every $\nu < \xi$. Let $\alpha_\xi \in \lambda - \bigcup\{D_{ij\nu} \mid i, j < n \text{ and } \nu < \xi\} - \{\alpha_\nu \mid \nu < \xi\}$. Let $D = \{\alpha_\nu \mid \nu < \lambda\}$; then $\{a_\alpha \mid \alpha \in D\}$ is a chain. Q.E.D.

COROLLARY 5.7. *If B is SC, then every subalgebra of B which is embeddable in an interval algebra has power $\leqslant \aleph_0$.*

PROOF. B does not contain uncountable chains or antichains, so the corollary follows from Theorem 5.6. (Alternatively 5.5 can be used.)[1]

Our last goal in this section is to prove, assuming CH, that the theory of BA's in Magidor-Malitz language L^2 is undecidable. (See definition in **[MM]**.)

Sierpinski assuming CH (but see also Bonnet **[B1]**) constructed a family $\{L_\alpha \mid \alpha < 2^{\aleph_1}\}$ of subsets of $\mathbf{R}$, such that if f is an order preserving or order reversing function, and $\mathrm{Dom}(f) \subseteq L_\alpha$, $\mathrm{Rng}(f) \subseteq L_\beta$ and $\alpha \neq \beta$, then $|\mathrm{Dom}(f)| \leqslant \aleph_0$. It is easy to see that if $\alpha \neq \beta$, then every linear ordering which is embeddable in both $B(L_\alpha)$ and $B(L_\beta)$ is of power $\leqslant \aleph_0$.

We now show how to interpret in the L^2-theory of BA's the first order theory of symmetric irreflexive relations. Let h: $\{\langle \sigma, \alpha \rangle \mid \sigma \subseteq \omega,\ |\sigma| = 2,\ \alpha < \aleph_1\} \to \aleph_1$ be a 1-1 function. Let $B_{\sigma,\alpha} = B(L_{h(\sigma,\alpha)})$, where $\{L_\alpha \mid \alpha < 2^{\aleph_1}\}$ is as mentioned above.

Let $M = \langle \beta, R \rangle$ be a structure, such that $\beta \leqslant \omega$ and R is an irreflexive symmetric relation on α.

We now define a BA B_M, in which M can be interpreted. For every $i \in \beta$, let $B_i = \Sigma\{B_{\{i,j\},\alpha} \mid \langle i, j \rangle \in R,\ \alpha < \aleph_1\}$ where Σ denotes the weakest direct product. Let $B_M = \Sigma_{i \in \beta} B_i$. Let $\varphi_0(x)$ be the formula in L^2 that says: "there is an uncountable family of pairwise disjoint nonzero subelements of x." Let

$$\varphi_1(x) \equiv \varphi_0(x) \wedge \forall y_1, y_2 \left([(y_1 \cup y_2 = x) \wedge (y_1 \cap y_2 = 0)] \to \bigvee_{i=1}^{2} \neg\varphi_0(y_i) \right).$$

Let $\varphi_2(x, y) = \varphi_1(x) \wedge \varphi_1(y) \wedge \neg\varphi_0(x \triangle y)$. It is clear that in B_M $\varphi_2(x, y)$ defines an equivalence relation E on $\{a \mid B_M \vDash \varphi_1[a]\}$, and for every equivalence class C of

E, there is a unique $i < \beta$ such that $1_{B_i} \in C$. Let us denote this equivalence class by C_i.

Let $\varphi_3(y_1, y_2)$ be the formula which says:

$$(\exists X)(|X| \geq \aleph_1 \wedge (\forall x_1 x_2 \in X)((x_1 \text{ and } x_2 \text{ are comparable}) \wedge[(x_1 = x_2) \vee ((x_1 \cap y_1 \neq x_2 \cap y_1 \wedge (x_1 \cap y_2 \neq x_2 \cap y_2))]).$$

$\varphi_3(y_1, y_2)$ says that there is an uncountable linear ordering which is embeddable in both $B \restriction y_1$ and in $B \restriction y_2$.

Let $\varphi_4(x_1, x_2)$ be the formula that says: $(\exists X)(|X| \geq \aleph_1 \wedge (\forall y_1 y_2 \in X)(y_1 = y_2 \vee (y_1 \cap y_2 = 0 \wedge \varphi_3(y_1 \cap x_1, y_1 \cap x_2))))$.

Then, for every $i, j \in \beta$ and for every $x_1 \in C_i$ and $x_2 \in C_j$: $B_M \models \varphi_4[x_1, x_2]$ iff $\langle i, j \rangle \in R$.

Conclusion 5.8. (CH) The first order theory of irreflexive symmetric relations is interpretable in the L^2-theory of BA's, and so the L^2-theory of BA's is undecidable.

6. Retractiveness of free products. We have already seen that $\diamondsuit_{\aleph_1}$ implies the existence of a retractive BA not embeddable in an interval algebra. The results of this section are motivated by the following open question.

Question 6.1. Does ZFC imply the existence of a retractive BA not embeddable in an interval algebra?

Throughout this section $B_1 * B_2$ denotes the free product of B_1 and B_2 and $\tilde{B}$ denotes the BA of finite and cofinite subsets of ω. Rotman [**R**] has proved that if B_1 is infinite and B_2 is uncountable, then $B_1 * B_2$ is not embeddable in an interval algebra.

We shall show that if L is a Suslin ordering, or if L is a Sierpinski set then $B(L) * \tilde{B}$ is retractive. In fact in Theorem 6.6 we shall find a necessary and sufficient condition on L that assures that $B(L) * \tilde{B}$ is retractive. We shall conclude that if there is a Suslin tree, or if MA holds, then there is a retractive BA not embeddable in an interval algebra.

In this section B is considered to be an ideal in itself; an ideal $I \neq B$ is called a proper ideal. If I is an ideal in B and A is a subalgebra of B we say that A is a retract of B relative to I if, for every $b \in B$, $|b/I \cap A| = 1$; an endomorphism h of B with kernel I such that $h^2 = h$ is called a retraction of B relative to I. Note that if h is a retraction of B relative to I, then $\mathrm{Rng}(h)$ is a retract of B relative to I. If I is an ideal in B and $a \in B$ let $I \restriction a = \{b \mid b \subseteq a \text{ and } b \in I\}$; clearly $I \restriction a$ is an ideal in $B \restriction a$.

LEMMA 6.2. (a) *Let* P_1 *denote the following property of a BA* B; (i) *B is retractive*; *and* (ii) *for every sequence* $\{I_i \mid i \in \omega\}$ *of ideals in* B, *there is a sequence* $\{\langle A_i, a_i, h_i\rangle \mid i \in \omega\}$ *such that* (1) $\{A_i \mid i \in \omega\}$ *is an increasing sequence of subalgebras of* B *whose union is* B; (2) $a_i \in A_i \cap I_i$; *and* (3) h_i *is a retract of* $B \restriction -a_i$ *relative to* $I_i \restriction -a_i$, *and* $h_i \restriction (A_i \restriction -a_i) = \mathrm{Id}$.

If B has property P_1 *then* $B * \tilde{B}$ *is retractive*.

(b) *Let* P_2 *be the following property of a BA* B: *for every sequence* $\{I_i \mid i \in \omega\}$ *of ideals in B there is an increasing sequence* $\{A_j \mid j \in \omega\}$ *of subalgebras of B whose union is B, and for every* $i, j \in \omega$ *there is* $a_{ij} \in I_i$ *such that for every* $b \in I_i \cap A_j$; $b \subseteq a_{ij}$.

If for some infinite B_1 $B * B_1$ *is retractive, then* P_2 *holds in* B.

PROOF. (a) Let $\tilde{B}$ have property P_1 and let $B = \bar{B} * \tilde{B}$. We denote by 1, $\bar{1}$ and $\tilde{1}$ the ones of B, $\bar{B}$ and $\tilde{B}$ respectively. Let $\{e_i \mid i \in \omega\}$ be a 1-1 enumeration of the atoms of $\tilde{B}$.

Let I be an ideal in B; we define the following ideals in $\bar{B}$: $I_i = \{b \in \bar{B} \mid b \cap e_i \in I\}$, $J = \{b \in \bar{B} \mid$ for some $n \in \omega$ $b \cap (\tilde{1} - \bigcup_{i<n} e_i) \in I\}$. Let $\{\langle A_i, a_i, h_i \rangle \mid i \in \omega\}$ be as assured by P_1 for the sequence of ideals $\{I_i \mid i \in \omega\}$. If $J = \bar{B}$, then by the retractiveness of $\bar{B}$ it is easy to find a retract of B relative to I. We thus assume that $J \neq \bar{B}$. Let C be a retract of $\bar{B}$ relative to J.

Let g_i be the endomorphism of $\bar{B}$ extending $\mathrm{Id} \restriction (\bar{B} \restriction a_i) \cup h_i$. For every $b \in \bar{B}$ let $b^* = \bigcup_{i \in \omega}(g_i(b) \cap e_i)$. A priori it is clear that $*$ is a homomorphism of $\bar{B}$ into the completion of B, but since $\{A_i \mid i \in \omega\}$ is an increasing sequence of subalgebras of $\bar{B}$ whose union in B, and by the definition of the g_i's, $*$ is really an embedding of $\bar{B}$ into B. Let $C^* = \{c^* \mid c \in C\}$, and let $C_i = \{b \cap e_i \mid b \in \mathrm{Rng}(h_i)\}$. Let A be the subalgebra of B generated by $C^* \cup \bigcup_{i \in \omega} C_i$. We claim that A is a retract of B relative to I.

We first show that for every $b \in B$ $|b/I \cap A| \geq 1$. Let $b \in B$; then for some $b_1 \in \bar{B}$ and $n \in \omega$

$$b \cap \left(\bar{1} \cap \left(\tilde{1} - \bigcup_{i<n} e_i\right)\right) = b_1 \cap \left(\tilde{1} - \bigcup_{i<n} e_i\right).$$

Let $c \in C$ be such that $c \triangle b_1 \in J$. Let m be chosen so that: $m \geq n$,

$$c^* \cap \left(\bar{1} \cap \left(\tilde{1} - \bigcup_{i<m} e_i\right)\right) = c \cap \left(\tilde{1} - \bigcup_{1<m} e_i\right)$$

and

$$(c \triangle b_1) \cap \left(\tilde{1} - \bigcup_{i<m} e_i\right) \in I.$$

Let $d = (c^* \triangle b) \cap (\bar{1} \cap \bigcup_{i<m} e_i)$; so for some $d_0, \dots, d_{m-1} \in \bar{B}$, $d = \bigcup_{i<m}(d_i \cap e_i)$. Let $d_i' = h_i(d_i - a_i)$ and let $d' = \bigcup_{i<m}(d_i' \cap e_i)$. It is easy to check that $d \triangle d' \in I$.

Clearly $d' \triangle c^* \in A$, and we show that $(d' \triangle c^*) \triangle b \in I$.

$$\begin{aligned}(d' \triangle c^*) \triangle b = &\left((d' \triangle c^* \triangle b) \cap \left(\bar{1} \cap \bigcup_{i<m} e_i\right)\right)\\ &\cup \left((d' \triangle c^* \triangle b) \cap \left(\bar{1} \cap \left(\tilde{1} - \bigcup_{1<m} e_i\right)\right)\right)\\ = &(d' \triangle d) \cup (c^* \triangle b) \in I.\end{aligned}$$

We have thus proved that for every $b \in B$ $|b/I \cap A| \geq 1$.

We now prove that $A \cap I = \{0\}$. It is easy to see that every element of A is a finite union of elements of the following forms: (1) $a \cap b$, where $a \in C^*$ and for some i $b \in C_i$; (2) $a - \bigcup_{j=1}^{k} c_j$, where $a \in C^*$, and there are $i_1, \dots, i_k$ such that $c_1 \in C_{i_1}, \dots, c_k \in C_{i_k}$. It suffices to show that every element of one of the above forms that belongs to I is equal to 0.

Suppose $a \in C^*$, $b \in C_i$ and $a \cap b \in I$. Hence there is $b' \in \mathrm{Rng}(h_i)$ such that $b = b' \cap e_i$, and for some $a' \in \mathrm{Rng}(g_i)$: $a \cap (\bar{1} \cap e_i) = a' \cap e_i$. By the definition of g_i $a' \cap b' \in \mathrm{Rng}(h_i)$. $a \cap b = (a' \cap b') \cap e_i$, so by the definition of I_i $a' \cap b' \in I_i$. But $\mathrm{Rng}(h_i) \cap I_i = \{0\}$, hence $a' \cap b' = 0$, and hence $a \cap b = 0$.

Let $a \in C^*$, $c_j \in C_{i_j}$, $j = 1, \ldots, k$, and $a - \bigcup_{j=1}^k c_j \in I$. Let $d \in \bar{B}$ be such that $a = d^*$, hence $d \in C$. Let $n > i_1, \ldots, i_k$ be such that

$$a \cap \left(\bar{1} \cap \left(\tilde{1} - \bigcup_{i<n} e_i\right)\right) = d \cap \left(\tilde{1} - \bigcup_{i<n} e_i\right);$$

hence

$$\left(a - \bigcup_{j=1}^{k} c_j\right) \cap \left(\bar{1} \cap \left(\tilde{1} - \bigcup_{i<n} e_i\right)\right) = d \cap \left(\tilde{1} - \bigcup_{i<n} e_i\right).$$

This means that $d \in J$. But $J \cap C = \{0\}$; hence $d = 0$, hence $d^* = 0$. Q.E.D.

(b) Suppose B_1 is infinite, and $\bar{B} * B_1$ is retractive; we show that P_2 holds in $\bar{B}$. Let $\{I_i \mid i \in \omega\}$ be a sequence of ideals in $\bar{B}$. Let $\{J_j \mid j \in \omega\}$ be an enumeration of $\{I_i \mid i \in \omega\}$ such that for every $i \in \omega$ $\{j \mid J_j = I_i\}$ is infinite. Let $\tilde{1}$ denote 1_{B_1}, and let $\{e_i \mid i \in \omega\}$ be a sequence of nonzero pairwise disjoint elements of B_1. Let $\tilde{B}$ be the subalgebra of B_1 generated by $\{e_i \mid i \in \omega\}$, and let I be the ideal in B generated by $\bigcup_{j\in\omega}\{b \cap e_j \mid b \in J_j\}$. Let h be a retraction of B relative to I. Let

$$A_j = \left\{b \in \bar{B} \mid h(b \cap \bar{1}) \cap \left(\bar{1} \cap \left(\tilde{1} - \bigcup_{k<j} e_k\right)\right) = b \cap \left(\tilde{1} - \bigcup_{k<j} e_k\right)\right\}.$$

It is easy to see that $\{A_j \mid j \in \omega\}$ is an increasing sequence of subalgebras whose union is $\bar{B}$.

Let $i, j \in \omega$. There is $k \geqslant j$ such that $J_k = I_i$. Let $a_{ij} \in \bar{B}$ be such that $h(\bar{1} \cap e_k) \cap (\bar{1} \cap e_k) = -a_{ij} \cap e_k$. Clearly $a_{ij} \in J_k = I_i$.

Let $a \in A_j \cap I_i$, then $a \in A_k \cap I_k$, hence $a \cap e_k \in I$, so

$$\begin{aligned} 0 = h(a \cap e_k) &= h(a \cap \bar{1}) \cap h(\bar{1} \cap e_k) \\ &\supseteq (a \cap e_k) \cap (-a_{ij} \cap e_k) = (a \cap -a_{ij}) \cap e_k. \end{aligned}$$

So $a \cap -a_{ij} = 0$. $a \subseteq a_{ij}$. Q.E.D.

The following lemma follows from the work of E. van Douwen [**D1**]. We bring it here for the sake of completeness.

LEMMA 6.3. *If $\bar{B}$ contains a nonprincipal noncountably generated ideal, and B' is infinite, then $\bar{B} * B'$ is not retractive.*

PROOF. Let $B = \bar{B} * B'$, and let 1, $\bar{1}$ and $1'$ denote the ones of B, $\bar{B}$, B' respectively. Let $\{b_i \mid i < \aleph_1\}$ be a sequence of elements of $\bar{B}$ such that for every $n \in \omega$ and $j_1, \ldots, j_n < j < \aleph_1$ $b_j \not\subseteq \bigcup_{m=1}^n b_{j_m}$. Let $\{a_k \mid k \in \omega\}$ be a strictly increasing sequence of elements of B', and let I be the ideal in B generated by $\{b_j \cap a_k \mid j < \aleph_1, k < \omega\}$. Suppose by contradiction there is a retraction $c \mapsto c^*$ of B relative to I.

For every $i < \aleph_1$ there is $k_i < \omega$ such that $(b_i \cap 1')^* \supseteq b_i \cap (1' - a_{k_i})$. Let k be such that $D \overset{\text{def}}{=} \{i \mid k_i = k\}$ is uncountable. Let $m > k$; then there are $i_1, \ldots, i_n < \aleph_1$ such that $(\bar{1} \cap a_m)^* \supseteq (\bar{1} - \bigcup_{l=1}^{n} b_{i_l}) \cap a_m$. Let $i > i_1, \ldots, i_n$; then

$$0 = (b_i \cap a_m)^* = (b_i \cap 1')^* \cap (\bar{1} \cap a_m)^*$$
$$\supseteq (b_i \cap (1' - a_k)) \cap \left(\left(\bar{1} - \bigcup_{l=1}^{n} b_{i_l} \right) \cap a_m \right)$$
$$= \left(b_i - \bigcup_{l=1}^{n} b_{i_l} \right) \cap (a_m - a_k) \neq 0.$$

A contradiction, so the lemma is proved. Q.E.D.

DEFINITION 6.4. (a) Let L be a linear ordering. An open interval tree (OIT) for L is a sequence $G = \{G_i \mid i \in \omega\}$ such that each G_i is a family $\{G_{ij} \mid j \in \alpha_i\}$ of pairwise disjoint open convex sets, and for every $i < k < \omega$ and $j \in \alpha_k$ there is $m \in \alpha_i$ such that $G_{kj} \subseteq G_{im}$.

(b) Let G be an OIT for L and $A \subseteq L$. A is called G-small if for every $i \in \omega$ $\{j \in \alpha_i \mid A \cap G_{ij} \neq \varnothing\}$ is finite. A is called $\sigma - G$-small if A is a countable union of G-small sets.

(c) L is thin if: (1) $|\{a \in L \mid a \text{ has a successor in } L\}| \leqslant \aleph_0$; and (2) for every OIT G, L is a $\sigma - G$-small.

PROPOSITION 6.5. *If L is thin, then* (a) *L is c.c.c., i.e. it does not contain an uncountable set of pairwise disjoint open sets*; (b) *every subset of L is thin*; *and* (c) *L is totally disconnected in its order topology.*

We leave the easy proof to the reader.

THEOREM 6.6. *$B(L) * \tilde{B}$ is retractive iff L is thin. If L is not thin then for every infinite B' $B(L) * B'$ is not retractive.*

PROOF. Suppose L is not thin. If L contains an uncountable set of elements which have a successor in L, then $B(L)$ has uncountably many atoms, and so the ideal of $B(L)$ generated by its atoms is not countably generated and so by 6.3 for every infinite B' $B(L) * B'$ is not retractive.

Suppose there is an OIT G such that L is not $\sigma - G$-small. Let $G = \{G_i \mid i \in \omega\}$ and $G_i = \{G_{ij} \mid j < \alpha_i \leqslant \omega\}$. We define ideals $\{I_i \mid i \in \omega\}$. I_i is the ideal in $B(L)$ generated by $\{(a, b] \mid \text{there is } j < \alpha_i \text{ such that } a, b \in G_{ij}\}$. Let $\{A_j \mid j \in \omega\}$ be an increasing sequence of subalgebras of $B(L)$ whose union is $B(L)$, and let $L_j = \{b \mid (-\infty, b] \in A_j\}$. Clearly $\bigcup_{j \in \omega} L_j = L$. We show that for some i and j: $\{k \mid |L_j \cap G_{ik}| \leqslant 2\}$ is infinite. If not, let

$$L'_j = L_j - \bigcup \{L_j - G_{ik} \mid i \in \omega, k \in \alpha_i \text{ and } |L_j \cap G_{ik}| = 1\}.$$

Clearly L'_j is G-small and $|L_j - L'_j| \leqslant \aleph_0$. Hence $|L - \bigcup_{j \in \omega} L'_j| \leqslant \aleph_0$. Let $\{a_i \mid i \in \omega\}$ be an enumeration of $L - \bigcup_{j \in \omega} L'_j$, and let $L''_j = L'_j \cup \{a_j\}$; then L''_j is G-small and $\bigcup_{j \in \omega} L''_j = L$, hence L is $\sigma - G$-small, a contradiction.

Let i, j be such that $\{k \mid |L_j \cap G_{ik}| \geqslant 2\}$ is infinite. We show that there is no $a \in I_i$ such that for every $b \in I_i \cap A_j$ $b \subseteq a$. Let $a \in I_i$; then there are $k_1, \ldots, k_m < \alpha_i$, $c_l, d_l \in G_{ik_l}$, $l = 1, \ldots, m$, such that $a = \bigcup_{l=1}^{m}(c_l, d_l]$. Let $k \neq k_1, \ldots, k_m$ $c, d \in G_{ik} \cap L_j$ and $c < d$. Then $(c, d] \not\subseteq a$ and $(c, d] \in A_j \cap I_i$. We have thus shown that $B(L)$ does not have property P_2, and hence $B(L) * B$ is not retractive unless B is finite.

We now assume that L is thin and prove that $B(L)$ has property P_1. In order to avoid some inessential technicalities, we restrict ourselves to the case when L is a dense linear ordering without endpoints; the proof for the general case does not involve additional ideas. Let $\{I_i \mid i \in \omega\}$ be a sequence of ideals in $B(L)$. For every $i \in \omega$ let $\{C_{ij} \mid j < \alpha_i\}$ be the set of convex subsets of $L \cup \{-\infty, \infty\}$ that determines I_i, that is: for every $j < \alpha_i$ $|C_{ij}| > 1$, the C_{ij}'s are pairwise disjoint and I_i is generated by $\bigcup_{j<\alpha_i}\{(a, b] \mid a, b \in C_{ij}\}$. Since L is c.c.c., for every i, α_i is countable, and so we assume that $\alpha_i \leqslant \omega$. Let H_{ij} be the open set obtained from C_{ij} by deleting its endpoints if they exist. Let $\mathbf{Z}$ denote the set of integers. For every i, j let $\{H_{ijz} \mid z \in \mathbf{Z}\}$ be a sequence of nonempty pairwise disjoint convex open sets whose union is H_{ij}, and such that if $z_1 < z_2$, then $H_{ijz_1} < H_{ijz_2}$. The existence of such a sequence follows from the total disconnectedness and the c.c.c.-ness of L. It is easy to find an OIT $G = \{G_i \mid i \in \omega\}$, $G_i = \{G_{ik} \mid k \in \omega\}$ such that for every i, j, z H_{ijz} is a union of members of G_i.

Let $A = \{a \mid$ there exist i, j such that $a = \min(C_{ij})$ or $a = \max(C_{ij})\}$, and let $\{a_i \mid i \in \omega\}$ be an enumeration of A. Let $\{L'_i \mid i \in \omega\}$ be an increasing sequence of G-small sets whose union is $(L \cup \{-\infty, \infty\}) - A$. It is easy to see that for every i, j, k, L'_k is bounded in H_{ij}, and by the G-smallness of L'_k for every i and k $\{j \mid H_{ij} \cap L'_k \neq \varnothing\}$ is finite. So there are finite subsets of L, $\{\sigma_i \mid i \in \omega\}$, such that for every i and j, if $(L'_i \cup \bigcup_{l\leqslant i}\sigma_l) \cap H_{ij} \neq \varnothing$, then $(L'_i \cup \bigcup_{l\leqslant i}\sigma_l) \cap H_{ij}$ has a minimum and a maximum. Let $L_i = L'_i \cup \bigcup_{l\leqslant i}\sigma_l \cup \{a_l \mid l \leqslant i\}$. The sequence $\{L_i \mid i \in \omega\}$ is thus an increasing sequence whose union is $L \cup \{-\infty, \infty\}$, and for every i: $\{j \mid L_i \cap C_{ij} \neq \varnothing\}$ is finite, and if $L_i \cap C_{ij} \neq \varnothing$, then $L_i \cap C_{ij}$ has a minimum and a maximum.

Let A_i be the subalgebra of $B(L)$ generated by $\{(c, d] \mid c, d \in L_i\}$. Clearly $\{A_i \mid i \in \omega\}$ is an increasing sequence whose union is $B(L)$. Let

$$b_i = \bigcup\left\{\left[\min(L_i \cap C_{ij}), \max(L_i \cap C_{ij})\right] \mid L_i \cap C_{ij} \neq \varnothing\right\}, \qquad b_i \in A_i \cap I_i,$$

and for every $a \in A_i \cap I_i$, $a \subseteq b_i$. For every i we now define a function

$$g_i\colon L \cup \{-\infty, \infty\} \to L \cup \{-\infty, \infty\} : g_i \restriction \left(L \cup \{-\infty, \infty\} - \bigcup_{j<\alpha_i} C_{ij}\right) = \mathrm{Id};$$

for every $j < \alpha_i$, if $L_i \cap C_{ij} = \varnothing$, let $c_{ij} \in C_{ij}$ and for every $c \in C_{ij}$ let $g_i(c) = c_{ij}$; and for every $j < \alpha_i$, if $C_{ij} \cap L_i \neq \varnothing$, let $c_{ij} = \min(C_{ij} \cap L)$ and $d_{ij} = \max(C_{ij} \cap L)$, let $g \restriction [c_{ij}, d_{ij}] = \mathrm{Id}$, $g_i(c) = c_{ij}$ for every $c_{ij} > c \in C_{ij}$, and $g_i(c) = d_{ij}$ for every $d_{ij} < c \in C_{ij}$.

Let $a_i = -b_i$. We define an endomorphism h_i of $B(L) \restriction a_i$: h_i is the unique endomorphism such that, for every $(c, d] \in B(L) \restriction a_i$, $h_i((c, d]) = (g_i(c), g_i(d))]$.

It is easy to see that h_i is well defined and that $\{\langle A_i, a_i, h_i\rangle \mid i \in \omega\}$ satisfies the requirements of P_1. Q.E.D.

An uncountable subset of **R** that intersects every measure zero set in a countable set is called a Sierpinski set. Recall that CH implies the existence of a Sierpinski set.

A c.c.c. linear ordering which does not contain an uncountable subset order isomorphic to a subset of **R** is called a Suslin ordering.

THEOREM 6.7. (a) *If L is a Suslin ordering, then L is thin.*

(b) *A Sierpinski set is thin.*

(c) *MA implies that every subset of* **R** *of cardinality less than $2^{\aleph_0}$ is thin, and that there is a thin subset of* **R** *of cardinality $2^{\aleph_0}$.*

PROOF. (a) Let L be a Suslin ordering. Let $G = \{G_i \mid i \in \omega\}$ be an OIT for L, where $G_i = \{G_{ij} \mid j < \alpha_i\}$. Since L is c.c.c., w.l.o.g. for every $i \in \omega$, $\alpha_i = \omega$.

Let us define the following equivalence relation on L: $x \sim y$, if for every i, j, $x \in G_{ij}$, iff $y \in G_{ij}$, and $x < G_{ij}$ iff $y < G_{ij}$. Clearly a set that intersects every equivalence class in at most one element is embeddable in **R**, and hence the number of equivalence classes is at most $\aleph_0$. Let $\{C_i \mid i \in \omega\}$ be an enumeration of all equivalence classes. Clearly each C_i is G-small $\bigcup_{i\in\omega} C_i = L$, so L is $\sigma - G$-small.

(b) Let L be a Sierpinski set, and let G be an OIT for L. It is easy to see that for every interval $[a, b]$ of **R** and every $\varepsilon > 0$, there is a G-small closed set F such that $\mu([a, b] - F) < \varepsilon$. Let F_i be such a set for the interval $[-i, i]$ and $\varepsilon = 1/i$. Clearly $\mu(\mathbf{R} - \bigcup_{i\in\omega} F_i) = 0$, and hence $|L - \bigcup_{i\in\omega} F_i| \leq \aleph_0$. Let $\{a_i \mid i \in \omega\} = L - \bigcup_{i\in\omega} F_i$, and let $C_i = (F_i \cap L) \cup \{a_i\}$; then C_i is G-small, and $\bigcup_{i\in\omega} C_i = L$. This proves (b).

(c) Suppose MA holds. Let $L \subseteq R$ and $|L| < 2^{\aleph_0}$, and let $G = \{G_i \mid i \in \omega\}$ be an OIT in **R**. We shall show that there is a family $\{F_i \mid i \in \omega\}$ of G-small closed sets such that: $\mu(\mathbf{R} - \bigcup_{i\in\omega} F_i) = 0$, and $L \subseteq \bigcup F_i$.

Let $G_i = \{G_{ij} \mid j \in \omega\}$, $i \in \omega$, and let $P_\omega(A) \overset{\text{def}}{=} \{B \subseteq A \mid |B| < \aleph_0\}$. We define a forcing notion $\langle P, \leq\rangle$. Every $p \in P$ has the following form: $\{\langle \sigma_i^p, F_i^p\rangle \mid i \in \omega\}$ where: (1) for every $i \in \omega$: F_i^p is a finite union of closed intervals and rays with rational endpoints, $\mu([-i, i] - F_i^p) < 1/i$, and $\sigma_i^p \in P_\omega(L \cap F_i^p)$; and (2) $\{i \mid F_i^p \neq \mathbf{R}\}$ is finite. $p \leq q$ if for every i $\sigma_i^p \subseteq \sigma_i^q$ and $F_i^q \subseteq F_i^p$.

It is easy to see that P is c.c.c. For every i, $j \in \omega$ let $D_{ij} = \{p \mid \{k \mid F_i^p \cap G_{jk} \neq \varnothing\}$ is finite$\}$; clearly D_{ij} is dense in P. For every $a \in L$ let $D_a = \{p \mid a \in \bigcup_{i\in\omega} \sigma_i^p\}$; clearly D_a is dense in P. Let U be a filter in P which intersects all the D_a's and all the D_{ij}'s, and let $F_i = \bigcap_{p\in U} F_i^p$; then $\{F_i \mid i \in \omega\}$ is as required. This proves the first part of (c).

We now prove the second part of (c). Suppose MA holds, and let $\{G^\alpha \mid \alpha < 2^{\aleph_0}\}$ be an enumeration of all OIT's in **R**. We define by induction on α a family $\{F_i^\alpha \mid i \in \omega\}$ of closed G-small sets and $a_\alpha \in \mathbf{R}$ such that: (1) $\mu(\mathbf{R} - \bigcup_{i\in\omega} F_i) = 0$, and (2) $a_\alpha \in \bigcap_{\beta\leq\alpha} F_i^\beta - \{a_\beta \mid \beta < \alpha\}$.

Suppose a_β, $\{F_i^\beta \mid i \in \omega\}$ has been defined for every $\beta < \alpha$. By the first part of (c), there is a family $\{F_i^\alpha \mid i \in \omega\}$ of closed G^α-small sets containing $\{a_\beta \mid \beta < \alpha\}$, and such that $\mu(\mathbf{R} - \bigcup_{i\in\omega} F_i^\alpha) = 0$. By MA $\bigcap_{\beta\leqslant\alpha}(\bigcup_{i\in\omega} F_i^\beta) - \{a_\beta \mid \beta < \alpha\} \neq \varnothing$; let a_α belong to this set. It is easy to see that $\{a_\alpha \mid \alpha < 2^{\aleph_0}\}$ is thin. Q.E.D.

We now make some observations about the nonretractiveness of some free products.

Observation 6.8. (a) Let B be an atomless BA such that: (1) $B - \{1\}$ is the union of countably many proper ideals (i.e. B is embeddable in $P(\omega)$): (2) If $\{A_i \mid i \in \omega\}$ is an increasing sequence of subalgebras whose union is B, then there is $i \in \omega$ and $b \in B - \{0\}$ such that $A_i \restriction b$ is dense in $B \restriction b$. *Then* B does not have property P_2.

In particular SC BA's do not have property P_2.

(b) Let $A \subseteq B_1, B_2$ be uncountable BA's; then $B_1 * B_2$ contains a noncountably generated ideal.

It thus follows that if B_1 is uncountable and B is infinite, then $B_1 * B_1 * B$ is not retractive.

(c) If $B \subseteq B(\mathbf{R})$ is uncountable then $B * B$ is not retractive.

Proof. (a) Let $\{I_i \mid i \in \omega\}$ be a sequence of dense ideals whose union is $B - \{1\}$; let $\{A_j \mid j \in \omega\}$ be an increasing sequence of subalgebras whose union is B. We show that there are no a_{ij}'s as required in P_2. Let $j \in \omega$ and $b \in B - \{0\}$ be such that $A_j \restriction b$ is dense in $B \restriction b$. Let $i \in \omega$ be such that $-b \in I_i$. Clearly there is no a_{ij} as required in P_2.

(b) Let $a \to \tilde{a}$ be an isomorphism between two copies A and $\tilde{A}$ of A. Let $A \subseteq B$ and $\tilde{A} \subseteq \tilde{B}$; we show that ideal I of $B * \tilde{B}$ generated by $C \overset{\text{def}}{=} \{a \cap -\tilde{a} \mid a \in A\}$ is not countably generated. If it were countably generated, then there had been a countable subset C_0 of C generating I, but this is impossible, since for every $a \in A$ $a \cap -\tilde{a}$ does not belong to the ideal generated by $C - \{a \cap -\tilde{a}\}$.

(c) We shall prove a little bit more than what was stated in (c). A linear ordering $<$ of a subset C of a BA B is called a pseudo-order of C relative to B, if for every $c_1, \ldots, c_r < c < d_1, \ldots, d_s$ in C $\bigcap_{i=1}^n d_i \not\subseteq c \not\subseteq \bigcup_{i=1}^r c_i$. In such a case we shall call C a pseudo-chain. Let $<_i$ be a pseudo-order of a subset C_i of a BA B_i, $i = 1, 2$. We say that C_1 is pseudo-isomorphic to C_2 if $\langle C_1, <_1\rangle \cong \langle C_2, <_2\rangle$.

We prove the following claims.

Claim 1. If $B \subseteq B(\mathbf{R})$ is uncountable, then B contains an uncountable pseudo-chain.

Claim 2. Let $C_i \subseteq B_i$, $i = 1, 2$, be pseudo-isomorphic pseudo-chains, and suppose that the subalgebra of B_i generated by C_i is embeddable in $P(\omega)$, $i = 1, 2$. Then $B_1 * B_2$ is not retractive.

Clearly (c) follows from Claims 1 and 2.

Claim 1 follows easily from the fact that $\mathbf{R}$ is separable. We prove Claim 2. Let $<$ be a pseudo-order of C_1 relative to B_1, and $<_2$ be a pseudo-order of C_2. Let $c \to \tilde{c}$ be an isomorphism between $\langle C_1, < \rangle$ and $\langle C_2, <_2\rangle$. Let I be the ideal in $B_1 * B_2$ generated by $D \overset{\text{def}}{=} \{c \cap -\tilde{d} \mid c, d \in C_1 \text{ and } c < d\}$. Our goal is to show that there is no subalgebra A of $B_1 * B_2$ such that for every $b \in B_1 * B_2$ $|A \cap b/I| = 1$.

Let $E = \{c \cap -\tilde{c} \mid c \in C_1\}$, we check that E/I is a set of pairwise disjoint nonzero elements in $B_1 * B_2/I$. Let $c, d \in C_1$ and $c < d$; then $(c \cap -\tilde{c}) \cap (d \cap -\tilde{d}) = (c \cap d) \cap -(\tilde{c} \cup \tilde{d}) \subseteq c \cap -\tilde{d} \in I$. Hence the elements of E are pairwise disjoint modulo I.

Let $c \in C_1$ and suppose by contradiction that $c \cap -\tilde{c} \subseteq \bigcup_{i=1}^{m}(c_i \cap -\tilde{d}_i)$ where $c_i < d_i \in C_1$. W.l.o.g., for every $1 \leqslant i \leqslant k$, $c_i < c$ and, for every $k < i \leqslant m$, $c \leqslant c_i$. Let $e = (c - \bigcup_{i=1}^{k} c_i) \cap (\bigcap_{i=k+1}^{m} \tilde{d}_i - \tilde{c})$. Clearly $0 \neq e \subseteq c \cap -\tilde{c}$ and for every $1 \leqslant i \leqslant m$, $e \cap (c_i \cap -\tilde{d}_i) = 0$. Hence $c \cap -\tilde{c} \not\subseteq \bigcup_{i=1}^{m}(c_i \cap -d_i)$.

Suppose by contradiction that there is a subalgebra $A \subseteq B_1 * B_2$ such that for every $b \in B_1 * B_2$ $|A \cap b/I| = 1$. For every $c \in C_1$ let $a_c \in A \cap (c \cap -\tilde{c})/I$. $\{a_c \mid c \in C_1\}$ has to be a set of pairwise disjoint nonzero elements in $B_1 * B_2$. Let B_i' be the subalgebra of B_i generated by C_i, $i = 1,2$. For every $c \in C_1$ there are $d_c^1, d_c^2 \in I$ such that $a_c = ((c \cap -\tilde{c}) - d_c^1) \cup d_c^2$. There is $d_c \in D$ such that $d_c^1 \subseteq d_c$, so $a_c \supseteq (c \cap -\tilde{c}) - d_c$; since $c \cap -\tilde{c}/I \neq 0$, $(c \cap -\tilde{c}) - d_c \neq 0$. Hence $\{(c \cap -\tilde{c}) - d_c \mid c \in C_1\}$ is a subset of $B_1' * B_2'$ consisting of pairwise disjoint nonzero elements. This is impossible since $B_1' * B_2'$ is embeddable in $P(\omega)$ and thus it is c.c.c. Q.E.D.

REMARKS. (a) It follows from a result of Nyikos that MA + $\neg$CH $\models$. If $B * B$ is retractive, then B is countable.

(b) E. van Douwen has proved that if $\langle S, < \rangle$ is a Suslin ordering then $B(S) * B(S)$ is not retractive.

(c) Question 37 in **[DMR]** remains open; our claim quoted in note 12 there had an error. For more information about retractiveness see **[DMR]**.

Let us state some open questions.

(1) Does ZFC imply that there is a retractive BA not embeddable in an interval algebra?

(2) Is it consistent with ZFC that **R** does not contain an uncountable thin set?[2]

(3) Is it consistent with ZFC that there are uncountable BA's B_1 and B_2 such that $B_1 * B_2$ is retractive? Does the above follow from ZFC? What is the answer if we require in addition that $B_1 = B_2$, or that B_1, B_2 are embeddable in $P(\omega)$ or both? What is the answer when we require that B_1, B_2 are interval algebras or embeddable in interval algebras or embeddable in $B(L)$, where $L = \mathbf{R}$, or L is a Suslin ordering?

(4) Is it consistent with, or does it follow from ZFC that every subalgebra of a retractive BA is retractive?

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[2] *Added in proof.* A. Miller answered this question positively.

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